

ON TWO-FACED FAMILIES OF NON-COMMUTATIVE RANDOM VARIABLES

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ABSTRACT. We demonstrate that the notions of bi-free independence and combinatorial-bi-free independence of two-faced families are equivalent using a diagrammatic view of bi-non-crossing partitions. These diagrams produce an operator model on a Fock space suitable for representing any two-faced family of non-commutative random variables. Furthermore, using a Kreweras complement on bi-non-crossing partitions we establish the expected formulas for the multiplicative convolution of a bi-free pair of two-faced families.

1. INTRODUCTION

Free probability for pairs of faces, or simply bi-free probability, was introduced by Voiculescu in [5] as a generalization of the notion of free probability to allow the simultaneous study of “left-handed” and “right-handed” variables. Prior to this work, the left and right actions were only considered separately. Voiculescu demonstrated that many results in free probability, such as the existence of the free cumulants and the free central limit theorem, have direct analogues in the bi-free setting. However, free independence is equivalent to a variety of computational conditions, such as vanishing alternating moments of centered variables, or vanishing mixed cumulants. It was shown in Proposition 5.6 of [5] that such computational conditions for bi-freeness exist as a collection of universal polynomials on the mixed moments of a bi-free pair of two-faced families, but their explicit formulas were unknown.

Seeking an alternate approach to bi-free probability, Mastnak and Nica in [1] defined the (ℓ, r) -cumulant functions, which they predicted to be the universal polynomials of Voiculescu. Such cumulant functions were defined by considering permutations applied to non-crossing diagrams. Taking inspiration from the free case, they defined a pair of two-faced families z' and z'' to be combinatorially-bi-free if all mixed cumulants are zero, and posed the question of whether their definition was equivalent to the definition of bi-free independence of Voiculescu.

In this paper, we will provide an affirmative answer to their question, demonstrating the equivalence of bi-free independence and combinatorial-bi-free independence. Analyzing [1], one can take a diagrammatic view of the desired partitions which is more natural to the study of two-faced families of non-commutative random variables. In Section 2, after some preliminaries, we introduce this view via the notion of bi-non-crossing partitions. Such partitions are designed to encapsulate information about whether a variable should be considered on the left or on the right. One main goal of this paper is to demonstrate that bi-non-crossing partitions play the same role in bi-free probability as non-crossing partitions play in free probability.

Following Speicher in [4], we introduce the incident algebra on bi-non-crossing partitions in Section 3. The algebra enables an analysis of left and right variables simultaneously, and provides a method of Möbius inversion. This allows us to directly obtain the bi-free cumulant functions.

In Section 4 we will prove our main theorem, Theorem 4.3.1, which demonstrates that the two notions of bi-free independence are equivalent. To do so, we analyze the action of operators on free product spaces as in [5] to obtain explicit descriptions of Voiculescu’s universal polynomials. We give equivalent formulae for these polynomials using the bi-non-crossing Möbius function.

Using the combinatorially-bi-free approach, we will develop further results. In Section 5 we will describe a multiplicative free convolution of two-faced families. By extending the Kreweras complement approach of [3] to bi-non-crossing diagrams, we show that the bi-free cumulants of a product of two-faced families can be written as a convolution of the individual bi-free cumulants.

Finally, in Section 6 we construct an operator model in the linear operators on a Fock space for a two-faced family of non-commutative random variables. This generalizes the model from [2] and provides a bi-free analogue of Voiculescu’s non-commutative R -series.

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2. PRELIMINARIES

2.1. Free probability for pairs of faces. Throughout, $z = ((z_i)_{i \in I}, (z_j)_{j \in J})$ will denote a two-faced family in a non-commutative probability space (\mathcal{A}, φ) with the left face indexed by I , the right face indexed by J , and I and J disjoint. We will also let z' and z'' be two-faced families, similarly indexed.

Recall that in [5], z' and z'' are said to be *bi-freely independent* (or simply *bi-free*) if there exists a free product $(\mathcal{X}, p, \xi) = (\mathcal{X}', p', \xi') * (\mathcal{X}'', p'', \xi'')$ of vector spaces with specified state-vectors and unital homomorphisms

$$\begin{aligned} l^\epsilon &: \mathbb{C} \langle z_i^\epsilon : i \in I \rangle \rightarrow \mathcal{L}(\mathcal{X}^\epsilon), & \text{and} \\ r^\epsilon &: \mathbb{C} \langle z_j^\epsilon : j \in J \rangle \rightarrow \mathcal{L}(\mathcal{X}^\epsilon), & \epsilon \in \{', ''\}, \end{aligned}$$

such that the two-faced families $T^\epsilon = ((\lambda^\epsilon \circ l^\epsilon(z_i^\epsilon))_{i \in I}, (\rho^\epsilon \circ r^\epsilon(z_j^\epsilon))_{j \in J})$ with $\epsilon \in \{', ''\}$ have the same joint distribution in $(\mathcal{L}(\mathcal{X}), \varphi)$ as z' and z'' . Here λ^ϵ and ρ^ϵ are the left and right representations of $\mathcal{L}(\mathcal{X}^\epsilon)$ in $\mathcal{L}(\mathcal{X})$ (cf. Section 1.9 in [5]). For $T \in \mathcal{L}(\mathcal{X}^\epsilon)$, we will often repress the ϵ notation on λ^ϵ , ρ^ϵ , and φ^ϵ (the state on $\mathcal{L}(\mathcal{X}^\epsilon)$ induced by p^ϵ) as it will be clear which is meant by noting which vector space T is defined on.

Given $\alpha : \{1, \dots, n\} \rightarrow I \sqcup J$, we will refer to the “ α -moment” of a two-faced family z :

$$\varphi_\alpha(z) := \varphi(z_{\alpha(1)} \cdots z_{\alpha(n)}).$$

It was shown in Theorem 5.7 of [5] that for each α there exists a universal polynomial R_α on indeterminates X_K indexed by non-empty subsets $K \subset \{1, \dots, n\}$ satisfying:

- (i) $R_\alpha = X_{\{1, \dots, n\}} + \tilde{R}_\alpha$, where \tilde{R}_α is a polynomial on indeterminates X_K indexed by non-empty strict subsets $K \subsetneq \{1, \dots, n\}$;
- (ii) R_α and \tilde{R}_α are homogeneous of degree n when X_K is given degree $|K|$; and
- (iii) if $R_\alpha(z)$ denotes R_α evaluated at $X_K = \varphi(z_{\alpha(k_1)} \cdots z_{\alpha(k_r)})$ with $K = \{k_1 < \cdots < k_r\}$, then

$$R_\alpha(z' + z'') = R_\alpha(z') + R_\alpha(z''),$$

when z' and z'' are bi-free two-faced families.

The number $R_\alpha(z)$ is called the α -cumulant of z . Property (iii) above is referred to as the *cumulant property*.

2.2. Combinatorial-bi-free independence. For consistency, we note the following definitions of Mastnak and Nica.

Definition 2.2.1 (Definition 5.2 of [1]). Let (\mathcal{A}, φ) be a non-commutative probability space. There exists a family of multilinear functionals

$$(\kappa_\chi : \mathcal{A}^n \rightarrow \mathbb{C})_{n \geq 1, \chi : \{1, \dots, n\} \rightarrow \{\ell, r\}}$$

which are uniquely determined by the requirement

$$\varphi(z_1 \cdots z_n) = \sum_{\pi \in \mathcal{P}(x)(n)} \left(\prod_{V \in \pi} \kappa_{\chi|_V}((z_1, \dots, z_n)|V) \right)$$

for every $n \geq 1$, $\chi \in \{\ell, r\}^n$, and $z_1, \dots, z_n \in \mathcal{A}$. These κ_χ 's will be called the (ℓ, r) -cumulant functionals of (\mathcal{A}, φ) .

Definition 2.2.2 ([1]). Let z' and z'' each be two-faced families in (\mathcal{A}, φ) . We say that z' and z'' are *combinatorially-bi-free* if

$$\kappa_\chi(z_{\alpha(1)}^{\epsilon_1}, \dots, z_{\alpha(n)}^{\epsilon_n}) = 0$$

whenever $\alpha : \{1, \dots, n\} \rightarrow I \sqcup J$, $\chi : \{1, \dots, n\} \rightarrow \{\ell, r\}$ is such that $\alpha^{-1}(I) = \chi^{-1}(\{\ell\})$, and $\epsilon \in \{', ''\}^n$ is non-constant.

Remark 2.2.3. Note that the condition $\alpha^{-1}(I) = \chi^{-1}(\{\ell\})$ completely determines χ and so we may denote

$$\kappa_\alpha(z) := \kappa_\chi(z_{\alpha(1)}, \dots, z_{\alpha(n)}).$$

Then if z' and z'' are combinatorially-bi-free, it is easy to see that

$$\kappa_\alpha(z' + z'') = \kappa_\alpha(z') + \kappa_\alpha(z'');$$

that is, κ_α has the cumulant property.

2.3. Partitions, ordering, and non-crossing partitions. A partition π is a set $\pi = \{V_1, \dots, V_k\}$, where V_1, \dots, V_k (called the *blocks of π*) are non-empty sets satisfying $V_i \cap V_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^k V_i = \{1, \dots, n\}$. We traditionally order the blocks of π so that $\min(V_1) < \dots < \min(V_k)$. Let $\mathcal{P}(n)$ denote the set of partitions of $\{1, \dots, n\}$.

For $\pi, \sigma \in \mathcal{P}(n)$ we say π is a *refinement* of σ and write $\pi \leq \sigma$ if every block of π is contained in a block of σ . This defines a partial ordering on $\mathcal{P}(n)$ with minimum and maximum elements

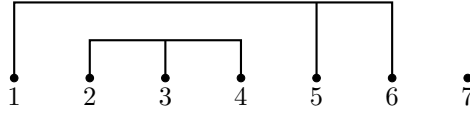
$$0_n := \{\{1\}, \dots, \{n\}\}, \quad 1_n := \{\{1, \dots, n\}\},$$

respectively. We will also consider the following action of the symmetric group S_n on $\mathcal{P}(n)$: if $\pi = \{V_1, \dots, V_k\} \in \mathcal{P}(n)$ and $s \in S_n$ then

$$s \cdot \pi = \{s(V_1), \dots, s(V_k)\} \in \mathcal{P}(n).$$

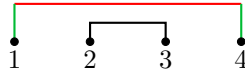
Observe that this action is order-preserving.

A partition $\pi \in \mathcal{P}(n)$ is said to be *non-crossing* if for any two distinct blocks $V = \{v_1 < \dots < v_r\}, W = \{w_1 < \dots < w_s\} \in \pi$ we have $v_l < w_1 < v_{l+1}$ if and only if $v_l < w_s < v_{l+1}$ ($l \in \{1, \dots, r-1\}$). The term “non-crossing” refers to the fact that any such partition can be represented as a non-crossing diagram. For example, the non-crossing partition $\{\{1, 5, 6\}, \{2, 3, 4\}, \{7\}\} \in \mathcal{P}(7)$ corresponds to the diagram



We denote set of non-crossing partitions in $\mathcal{P}(n)$ by $NC(n)$.

The horizontal segments connected the nodes of a block $V \in \pi$ will be referred to as the *spine of V* , and the segments connecting the nodes to the spine of V will be referred to as the *ribs of V* . In the following diagrammatic representation of $\{\{1, 4\}, \{2, 3\}\} \in NC(4)$ we have highlighted the spine of $\{1, 4\}$ in red and its ribs in green:



For a singleton block $V \in \pi$, $|V| = 1$, the spine of V will simply refer to the corresponding node itself.

2.4. Bi-non-crossing partitions. For $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$ we let $\{i_1 < \dots < i_p\} = \alpha^{-1}(I)$ and $\{j_1 < \dots < j_{n-p}\} = \alpha^{-1}(J)$ and consider $s_\alpha \in S_n$ defined by

$$s_\alpha(k) = \begin{cases} i_k & \text{if } k \leq p \\ j_{n+1-k} & \text{if } k > p \end{cases}.$$

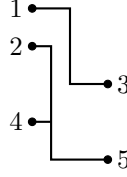
We say a partition $\pi \in \mathcal{P}(n)$ is *bi-non-crossing* (with respect to α) if $s_\alpha^{-1} \cdot \pi \in NC(n)$. We denote the set of such partitions by $BNC(\alpha)$. The minimum and maximum elements of $BNC(\alpha)$ are given by $0_\alpha := s_\alpha \cdot 0_n$ and $1_\alpha := s_\alpha \cdot 1_n$, respectively.

To each partition $\pi \in BNC(\alpha)$ we can associate a “bi-non-crossing diagram” as follows. For each $k = 1, \dots, n$ place a node labeled k at the position $(-1, n - k)$ if $\alpha(k) \in I$ and at the position $(1, n - k)$ if $\alpha(k) \in J$. Connect nodes whose labels form a block of π similar to how one would for the diagrams associated to $NC(n)$, except now the spines of blocks are vertically oriented and the ribs extend horizontally from the spine to the left or right, emphasizing the left-right nature of a two-faced family.

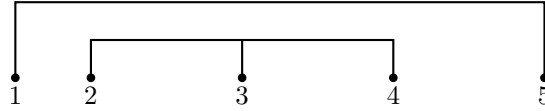
Example 2.4.1. If $\alpha^{-1}(I) = \{1, 2, 4\}$, $\alpha^{-1}(J) = \{3, 5\}$, and

$$\pi = \{\{1, 3\}, \{2, 4, 5\}\} = s_\alpha \cdot \{\{1, 5\}, \{2, 3, 4\}\},$$

then the bi-non-crossing diagram associated to π is



That the diagram can always be drawn to be non-crossing is easily seen through its relationship to the diagram of $s_\alpha^{-1} \cdot \pi \in NC(n)$. Indeed, rotate the line $x = -1$ counter-clockwise a quarter turn about the point $(-1, 0)$, rotate the line $x = 1$ clockwise a quarter turn about the point $(1, 0)$, and adjust the spines and ribs so that they remain connected. Then after relabeling node k as $s_\alpha^{-1}(k)$ the resulting diagram is precisely the one associated to $s_\alpha^{-1} \cdot \pi$ as an element of $NC(n)$ (modulo some extra space between the nodes). Performing this operation to the above diagram yields



Conversely given the diagram corresponding to $\sigma \in NC(n)$ we obtain the diagram for $\pi = s_\alpha \cdot \sigma$ as follows. Initially, the nodes occupy positions $(1, 0), \dots, (n, 0)$, so we first widen the space between nodes so that node k now occupies position $(s_\alpha(k), 0)$ if $k \leq |\alpha^{-1}(I)|$ and position $(n + 1 - s_\alpha(k), 0)$ if $k > |\alpha^{-1}(I)|$. Given the definition of s_α , it is clear that this does not change the order of the nodes. Next, we rotate the segment from $(1, 0)$ to $(n, 0)$ clockwise a quarter turn about $(n, 0)$, we rotate the segment from $(n + 1, 0)$ to $(2n, 0)$ counter-clockwise a quarter turn about $(n + 1, 0)$, and homotopically vary the spines and ribs so that they remain connected. Relabeling node k as node $s_\alpha(k)$ then yields the diagram corresponding to π .

Remark 2.4.2. Given $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$, define $\chi \in \{\ell, r\}^n$ by $\chi_k = \ell$ if $\alpha(k) \in I$ and $\chi_k = r$ if $\alpha(k) \in J$. Then $BNC(\alpha)$ is precisely the class of partitions $\mathcal{P}^{(\chi)}(n)$ defined in [1] since s_α defined above is exactly the permutation σ_χ used to define the class $\mathcal{P}^{(\chi)}(n)$. Moreover, the notation $BNC(\alpha)$ suggests that the lattice of partitions depends on α more than it actually does. In fact, if $\beta: \{1, \dots, n\} \rightarrow I \sqcup J$ is such that $\beta(j)$ and $\alpha(j)$ are in the same face for each $j = 1, \dots, N$, then $BNC(\alpha) = BNC(\beta)$. Because of this we may write $BNC(\chi)$ for $BNC(\alpha)$. In order to emphasize the diagrammatic viewpoint pervading this paper, we will continue to use the alternate notation of $BNC(\alpha)$ for this class of partitions.

2.5. Shaded bi-non-crossing diagrams and partitions. Let z' and z'' be a bi-free pair of two-faced families. Let $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ and $\epsilon \in \{', ''\}^n$. We recursively define a collection of diagrams $LR(\chi, \epsilon)$. For $n = 1$, $LR(\chi, \epsilon)$ consists of two parallel, vertical, transparent segments with a single node on the left segment if $\chi(1) = \ell$ or a single node on the right segment if $\chi(1) = r$. We assign a shade to $'$ and $''$ and shade this node the shade associated to ϵ_1 . Then either this node remains isolated or a rib and spine of the node's shade are drawn connecting to the top gap between the two segments.

For $n > 1$ we define $LR(\chi, \epsilon)$ as follows. Let $\chi_0 = \chi|_{\{2, \dots, n\}}$ and $\epsilon_0 = (\epsilon_2, \dots, \epsilon_n)$. Then a diagram of $LR(\chi, \epsilon)$ is an extension of a diagram $D \in LR(\chi_0, \epsilon_0)$: place an additional ϵ_1 -shaded node p above D , on the left if $\chi(1) = \ell$ and on the right otherwise. Extend any spines from D to the new top gap. If at least one spine was extended and the one nearest p shares its shade, then connect it to p with a rib and optionally terminate the spine at p . Otherwise, either connect p with a rib to a new spine extending to the top gap or leave p isolated.

Given its impact on the diagrams, we refer to $\epsilon \in \{', ''\}^n$ as a *choice of shading* or simply a *shading*.

Note that each diagram in $LR(\chi, \epsilon)$ is created from a unique diagram in $LR(\chi_0, \epsilon_0)$, which we can recover by simply erasing the top portion of the diagram. Also, these rules imply that among the chords extending to the top gap, adjacent chords will always be of differing shades. We take the convention that the nodes are labeled numerically from top to bottom.

For $0 \leq k \leq n$, let $LR_k(\chi, \epsilon) \subseteq LR(\chi, \epsilon)$ consist of those diagrams with precisely k chords extending to the top gap. Then $LR(\chi, \epsilon) = \bigcup_k LR_k(\chi, \epsilon)$.

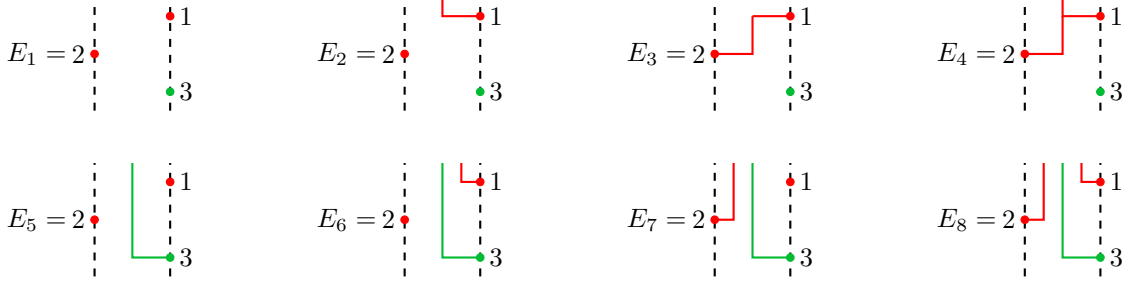
We consider a few examples. In each example, we assign the shade red to $'$ and the shade green to $''$ and have a dashed line in place of the normally transparent left and right segments.

Example 2.5.1. Consider $\chi = (\ell, r)$ and $\epsilon = (', '')$. Then $LR(\chi, \epsilon)$ consists of the following diagrams:



Also $LR_0(\chi, \epsilon) = \{D_1\}$, $LR_1(\chi, \epsilon) = \{D_2, D_3\}$, and $LR_2(\chi, \epsilon) = \{D_4\}$.

Example 2.5.2. For a slightly more robust example we consider $\chi = (r, \ell, r)$ and $\epsilon = (', ', '')$. Then $LR(\chi, \epsilon)$ consists of the following diagrams:



Observe in terms of the recursive construction of $LR(\chi, \epsilon)$, the diagram D_k , $k = 1, 2, 3, 4$ from Example 2.5.1 creates diagrams E_{2k-1} and E_{2k} in the present example.

For fixed χ and ϵ we note that each $D \in LR_0(\chi, \epsilon)$ can be associated to a partition $\pi \in \mathcal{P}(n)$ by forming blocks according to which nodes are connected via chords in the diagram. Since $D \in LR_0(\chi, \epsilon)$ is completely determined by the connections between nodes, distinct diagrams yield distinct partitions. Moreover, if $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$ and we define χ^α by $\chi^\alpha(k) = \ell$ if $\alpha(k) \in I$ and $\chi^\alpha(k) = r$ if $\alpha(k) \in J$ then the partitions we obtain from $LR_0(\chi^\alpha, \epsilon)$ are elements of $BNC(\alpha)$. We denote by $BNC(\alpha, \epsilon)$ the partitions obtained from the diagrams in $LR_0(\chi^\alpha, \epsilon)$. It is not hard to see that given the diagram associated to some $\pi \in BNC(\alpha)$, there exists some shading ϵ such that $\pi \in BNC(\alpha, \epsilon)$. It then follows that

$$BNC(\alpha) = \bigcup_{\epsilon \in \{', ''\}^n} BNC(\alpha, \epsilon)$$

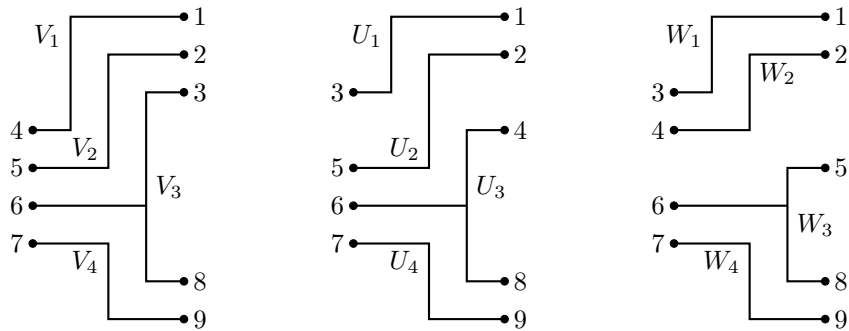
As with $BNC(\alpha)$, we may denote $BNC(\alpha, \epsilon)$ by $BNC(\chi, \epsilon)$ when $\chi = \chi^\alpha$.

Definition 2.5.3. Suppose that V and W are blocks of some $\pi \in BNC(\chi)$. Then V and W are said to be *piled* if $\max(\min(V), \min(W)) \leq \min(\max(V), \max(W))$. In terms of the diagram corresponding to π , the spines of V and W are not entirely above or below each other; there is some horizontal level at which both are present.

Given blocks V and W , a third block U *separates* V from W if it is piled with both, and its spine lies between the spines of V and W . Note that V and W need not be piled with each other to have a separator. Equivalently, U is piled with both V and W , and there are $j, k \in U$ such that $s_\alpha^{-1}(V) \subseteq [s_\alpha^{-1}(j), s_\alpha^{-1}(k)]$ and $s_\alpha^{-1}(W) \cap [s_\alpha^{-1}(j), s_\alpha^{-1}(k)] = \emptyset$, or vice versa. Given any three piled blocks, one always separates the other two.

Finally, piled blocks V and W are said to be *tangled* if there is no block which separates them.

Example 2.5.4. Consider the following diagrams.



In the first diagram, V_2 separates V_1 from V_3 , and all three are piled with one another. In the second diagram, U_2 still separates U_1 and U_3 , but U_1 and U_3 are not piled with each other. In the third diagram, there are no separators.

Definition 2.5.5. Suppose $\pi, \sigma \in BNC(\chi)$ are such that $\pi \leq \sigma$. We say π is a *lateral refinement* of σ and write $\pi \leq_{\text{lat}} \sigma$ if no two piled blocks in π are contained in the same block of σ .

Lateral refinements correspond to making lateral “cuts” along the spines of blocks of π , between their ribs.

In the notation of Example 2.5.2, E_1 is a lateral refinement of E_3 made by cutting the block $\{1, 2\}$ in between node 1 and node 2.

Lemma 2.5.6. *If $\pi \in BNC(\chi, \epsilon)$ then piled blocks of the same shade in π must be separated. Consequently, if $\sigma \in BNC(\alpha, \epsilon)$ and $\pi \leq \sigma$ then $\pi \leq_{\text{lat}} \sigma$.*

Proof. Suppose V_1 and V_2 are piled blocks in $\pi \in BNC(\chi, \epsilon)$ which have the same shade. Without loss of generality, $k := \max(V_2) < \max(V_1)$. In the construction of the diagram generating π , when node k is placed the nearest spine must be of a different shade as k begins a new spine. In particular, this spine sits between the spines of V_1 and V_2 , and so its block is a separator.

If two blocks of the same in π are piled, the above argument demonstrates that they are separated by a block of a different shade and so can't be joined in σ . \square

3. THE INCIDENT ALGEBRA ON BI-NON-CROSSING PARTITIONS

Definition 3.0.1. The lattice of bi-non-crossing partitions is

$$BNC := \bigcup_{n \geq 1} \bigcup_{\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}} BNC(\chi)$$

where the lattice structure on $BNC(\chi)$ is as above.

Given any lattice, there is an algebra of functions associated to the lattice.

Definition 3.0.2. The incident algebra on BNC , denoted $IA(BNC)$, is all functions of the form

$$f : \bigcup_{n \geq 1} \left(\bigcup_{\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}} BNC(\chi) \times BNC(\chi) \right) \rightarrow \mathbb{C}$$

such that $f(\pi, \sigma) = 0$ if $\pi \not\leq \sigma$ equipped with pointwise addition and a convolution product defined by

$$(f * g)(\pi, \sigma) = \sum_{\pi \leq \rho \leq \sigma} f(\pi, \rho) g(\rho, \sigma)$$

for all $\pi, \sigma \in BNC(\chi)$ and $f, g \in IA(BNC)$.

It is elementary to show that $IA(BNC)$ is an algebra and thus $(f * g) * h = f * (g * h)$.

3.1. Multiplicative functions on the incident algebra. In order to construct the notion of multiplicative functions on BNC , it is necessary to identify the lattice structure of an interval as a product of full intervals.

Proposition 3.1.1. *Let $\pi, \sigma \in BNC(\chi)$ be such that $\pi \leq \sigma$. The interval*

$$[\pi, \sigma] = \{\rho \in BNC(\chi) \mid \pi \leq \rho \leq \sigma\}$$

can be associated to a product of full lattices

$$\prod_{j=1}^k BNC(\beta_k)$$

for some $\beta_k : \{1, \dots, m_k\} \rightarrow \{\ell, r\}$ so that the lattice structure is preserved.

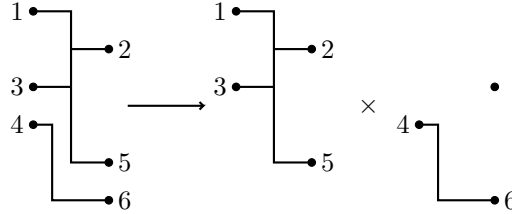
Proof. The idea behind the decomposition is to take π and σ , view π and σ as elements of $NC(n)$ by applying s_χ^{-1} , and using the decomposition of intervals in $NC(n)$ given in Proposition 1 of [4] while maintaining the notion of left and right nodes.

First write $\sigma = \{W_1, \dots, W_k\}$. Let π_j and σ_j be the restrictions of π and σ to W_j . Then we decompose $[\pi, \sigma]$ into

$$\prod_{j=1}^k [\pi_j, \sigma_j].$$

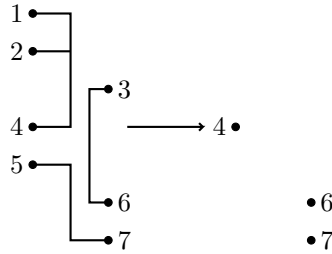
Note each σ_j is a full bi-non-crossing partition corresponding to some $\gamma_j : \{1, \dots, n_j\} \rightarrow \{\ell, r\}$ so one may reduce to intervals of the form $[\pi, 1_\chi]$.

For a fixed $\chi : \{1, \dots, n\} \rightarrow \{\ell, r\}$, a modification to the recursive argument of Proposition 1 of [4] under the identification of $BNC(\chi)$ with $NC(n)$ will be described. First, viewing $\pi \in NC(n)$, examine whether π has a block $V = \{k_1 < k_2 < \dots < k_m\}$ containing non-consecutive elements; that is, there exists an index t such that $k_t + 1 \neq k_{t+1}$. If so, the recursive argument of Proposition 1 of [4] would decompose $[\pi, 1_\chi]$ into the product of two intervals (removing any trivial intervals that occur): one corresponding to taking $[\pi, 1_\chi]$ and removing all nodes strictly between k_t and k_{t+1} ; and the other corresponding to taking only the nodes strictly between k_t and k_{t+1} and adding an isolated node on the left. The only change made to accommodate BNC is that the isolated node for the second interval should be added to the top left of the bi-non-crossing diagram if the lower of the two nodes of the original diagram corresponding to k_t and k_{t+1} is on the left and otherwise on the top right. For example:



Note that the first term in the product will be ignored as it is a full partition.

This recursive process eventually terminates leaving only partitions π such that the blocks of $\sigma_\chi^{-1} \cdot \pi$ are intervals. For such a bi-non-crossing partition, we associate the zero bi-non-crossing partition corresponding to keeping only the lowest node of each block. For example:



Thus we have reduced $[\pi, \sigma]$ to products of full lattices in BNC . □

Note that as in Proposition 1 of [4] we make no claim that this association is unique. However, this ambiguity does affect the following computations.

Definition 3.1.2. A function $f \in IA(BNC)$ is said to be *multiplicative* if whenever $\pi, \sigma \in BNC(\chi)$ are such that

$$[\pi, \sigma] \leftrightarrow \prod_{j=1}^k BNC(\beta_k)$$

for some $\beta_k : \{1, \dots, m_k\} \rightarrow \{\ell, r\}$, then

$$f(\pi, \sigma) = \prod_{j=1}^k f(0_{\beta_k}, 1_{\beta_k}).$$

For a multiplicative function $f \in IA(BNC)$, we will call the collection $\{f([0_\chi, 1_\chi]) \mid n \geq 1, \chi : \{1, \dots, n\} \rightarrow \{\ell, r\}\} \subseteq \mathbb{C}$ the *multiplicative net* associated to f . Note that for any net $\Lambda = \{a_\chi \mid n \geq 1, \chi : \{1, \dots, n\} \rightarrow \{\ell, r\}\} \subseteq \mathbb{C}$ there is precisely one multiplicative function f with multiplicative sequence Λ .

Lemma 3.1.3. *If $f, g \in IA(BNC)$ are multiplicative, then $f * g$ is multiplicative.*

See Proposition 2 of [4] for a proof of the above.

Remark 3.1.4. There are three special multiplicative functions to consider; namely

$$\delta_{BNC}(\pi, \sigma) = \begin{cases} 1 & \text{if } \pi = \sigma \\ 0 & \text{otherwise} \end{cases}$$

which is called the delta function on BNC and is the identity element in $IA(BNC)$,

$$\zeta_{BNC}(\pi, \sigma) = \begin{cases} 1 & \text{if } \pi \leq \sigma \\ 0 & \text{otherwise} \end{cases}$$

which is called the zeta function on BNC , and μ_{BNC} which is called the Möbius function on BNC which is defined such that

$$\mu_{BNC} * \zeta_{BNC} = \zeta_{BNC} * \mu_{BNC} = \delta_{BNC}$$

(as it is clear that ζ_{BNC} a left and right (and thereby a two-sided) inverse can be recursively defined). It is clear that δ_{BNC} is multiplicative with $\delta_{BNC}(0_\chi, 1_\chi)$ being one if $n = 1$ and zero otherwise, and ζ_{BNC} is multiplicative with $\zeta_{BNC}(0_\chi, 1_\chi) = 1$ for all χ . In addition, one can verify that μ_{BNC} is multiplicative and for any $\pi, \sigma \in BNC(\chi)$

$$\mu_{BNC}(\pi, \sigma) = \mu(s_\chi^{-1} \cdot \pi, s_\chi^{-1} \cdot \sigma),$$

where μ is the Möbius function in [4]. In addition, if $\pi, \sigma \in BNC(\chi)$ and we view π and σ as elements of $NC(n)$ as in the first paragraph of Proposition 3.1.1, one obtains by construction.

Remark 3.1.5. To consolidate the above with Subsection 2.2, for T_1, \dots, T_n in a non-commutative probability space (\mathcal{A}, φ) and $\pi \in BNC(\chi)$ where $\chi : \{1, \dots, n\} \rightarrow \{\ell, r\}$ and $V_t = \{k_{t,1} < \dots < k_{t,m_t}\}$ for $t \in \{1, \dots, k\}$ being the blocks of π , we define

$$\varphi_\pi(T_1, \dots, T_n) := \prod_{t=1}^k \varphi(T_{k_{t,1}} \cdots T_{k_{t,m_t}})$$

and

$$\kappa_\pi(T_1, \dots, T_n) := \sum_{\sigma \in BNC(\chi), \sigma \leq \pi} \varphi_\sigma(T_1, \dots, T_n) \mu_{BNC}(\sigma, \pi).$$

Then, as in [4], one can show that

$$\kappa_\pi(T_1, \dots, T_n) = \prod_{t=1}^k \kappa_{\pi|_{V_t}}(T_{k_{t,1}} \cdots T_{k_{t,m_t}})$$

where $\kappa_{\pi|_{V_t}}$ should be thought of as the (single block) partition induced by the block V_t of π , and

$$\varphi(T_1 \cdots T_n) = \sum_{\pi \in BNC(\chi)} \kappa_\pi(T_1, \dots, T_n).$$

In particular, $\kappa_{1_\chi} = \kappa_\chi$ are the bi-free cumulant functions of Definition 5.2 of [1].

For a two-faced family $z = ((z_i)_{i \in I}, (z_j)_{j \in J})$, $\alpha : \{1, \dots, n\} \rightarrow I \sqcup J$, and $\pi \in BNC(\alpha)$ we denote

$$\varphi_\pi(z) := \varphi_\pi(z_{\alpha(1)}, \dots, z_{\alpha(n)}) \quad \text{and} \quad \kappa_\pi(z) := \kappa_\pi(z_{\alpha(1)}, \dots, z_{\alpha(n)}).$$

In particular, $\varphi_{1_\alpha}(z) = \varphi_\alpha(z)$ and $\kappa_{1_\alpha}(z) = \kappa_\alpha(z)$. When the faces consist of a single element each, say z_ℓ and z_r , we define the above quantities for $\chi : \{1, \dots, n\} \rightarrow \{\ell, r\}$ replacing α . In this case we let $m_z, \kappa_z \in IA(BNC)$ be the multiplicative functions with multiplicative nets $(\varphi_\chi(z))_\chi$ and $(\kappa_\chi(z))_\chi$, respectively. We call m_z the *moment function* and κ_z the *bi-free cumulant function*. Thus the formulae $m_z * \mu_{BNC} = \kappa_z$ and $\kappa_z * \zeta_{BNC} = m_z$ are obtained.

4. UNIFYING BI-FREE INDEPENDENCE

4.1. Computing bi-free moments. We will demonstrate how the partitions of $BNC(\chi, \epsilon)$ may be used to compute joint moments of a bi-free pair of two-faced families.

Fix $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ and a shading $\epsilon \in \{', ''\}^n$, and let $T_k \in \mathcal{L}(\mathcal{X}^{\epsilon_k})$. Given $D \in LR(\chi, \epsilon)$, we will assign a vector weight $\psi(D; T_1, \dots, T_n) \in \mathcal{X}$ to D . Define $\mu \in \{\lambda, \rho\}^n$ by $\mu_j = \lambda$ if $\chi(j) = \ell$ and $\mu_j = \rho$ if $\chi(j) = r$. Let $V = \{k_1 < \dots < k_r\}$ be a block in D and let $\epsilon(V) := \epsilon_{k_1} = \dots = \epsilon_{k_r}$. If the spine of V is not connected to the top gap then V contributes a scalar factor of

$$\psi(V; T_1, \dots, T_n) := \psi^{\epsilon(V)} \left(T_{k_1} (1 - p^{\epsilon(V)}) T_{k_2} \dots (1 - p^{\epsilon(V)}) T_{k_r} \xi^{\epsilon(V)} \right)$$

to $\psi(D; T_1, \dots, T_n)$. If the spine does reach the top gap then it contributes a vector factor of

$$\psi(V; T_1, \dots, T_n) := (1 - p^{\epsilon(V)}) T_{k_1} (1 - p^{\epsilon(V)}) T_{k_2} \dots (1 - p^{\epsilon(V)}) T_{k_r} \xi^{\epsilon(V)}.$$

Then $\psi(D; T_1, \dots, T_n)$ is the product of the scalar factors and the tensor product of the vector factors where the order in the tensor product is determined by the left to right order of the spines reaching the top gap. If all contributions are scalar factors then we multiply this with the state-vector ξ , thinking of it as the “empty tensor word.”

Recalling Example 2.5.2, we see that

$$\begin{aligned} \psi(E_3; T_1, T_2, T_3) &= \psi'(T_1(1 - p')T_2\xi')\psi''(T_3\xi'')\xi, & \text{while} \\ \psi(E_8; T_1, T_2, T_3) &= (1 - p')T_2\xi' \otimes (1 - p'')T_3\xi'' \otimes (1 - p')T_1\xi' \end{aligned}$$

Let $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ and let $\pi \in BNC(\chi, \epsilon)$.

Proposition 4.1.1. *Fix $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ and a shading $\epsilon \in \{', ''\}^n$. Let $\mu \in \{\lambda, \rho\}^n$ be as above. If $T_j \in \mathcal{L}(\mathcal{X}^{\epsilon_j})$ for $j = 1, \dots, n$, then following formula holds:*

$$\mu_1(T_1) \cdots \mu_n(T_n) \xi = \sum_{D \in LR(\chi, \epsilon)} \psi(D; T_1, \dots, T_n). \quad (1)$$

Moreover,

$$\varphi(\mu_1(T_1) \cdots \mu_n(T_n)) = \sum_{\pi \in BNC(\chi)} \left[\sum_{\substack{\sigma \in BNC(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi}} (-1)^{|\pi| - |\sigma|} \right] \varphi_\pi(T_1, \dots, T_n) \quad (2)$$

Proof. We establish (1) via induction on n . The base case is clear, so we assume the formula holds for $n - 1$ operators and apply it as

$$\mu_2(T_2) \cdots \mu_n(T_n) \xi = \sum_{D \in LR(\chi_0, \epsilon_0)} \psi(D; T_2, \dots, T_n),$$

where $\chi_0 = \chi|_{\{2, \dots, n\}}$ and $\epsilon_0 = (\epsilon_2, \dots, \epsilon_n)$. Fix a $D \in LR(\chi_0, \epsilon_0)$ and assume $\mu_1 = \lambda$. Either there is a leftmost spine in D of the shade ϵ_1 reaching the top gap, or there is not (meaning either the nearest spine is the wrong shade or that D has no spines reaching the top gap). In the former case, writing $\psi(D; T_2, \dots, T_n)$ as $x_1 \otimes \dots \otimes x_m$ this implies $x_1 \in \mathcal{X}^{\epsilon_1}$. Hence

$$\begin{aligned} \lambda(T_1)x_1 \otimes \dots \otimes x_m &= \psi(T_1(1 - p^{\epsilon_1})x_1)x_2 \otimes \dots \otimes x_m + (1 - p^{\epsilon_1})T_1(1 - p^{\epsilon_1})x_1 \otimes x_2 \otimes \dots \otimes x_m \\ &= \psi(D_1; T_1, \dots, T_n) + \psi(D_2; T_1, \dots, T_n), \end{aligned}$$

where $D_1, D_2 \in LR(\chi, \epsilon)$ are the diagrams constructed from D by adding rib and, respectively, terminating the leftmost spine in D at the new top node or extending the leftmost spine in D .

If there is no leftmost spine of the same shade as ϵ_1 then $\psi(D; T_2, \dots, T_n)$ can be written in the same way as before except $x_1 \notin \mathcal{X}^{\epsilon_1}$ (if D has no spines reaching the top gap then this is simply a scalar multiple of ξ). Hence

$$\begin{aligned} \lambda(T_1)x_1 \otimes \dots \otimes x_m &= \psi^{\epsilon_1}(T_1\xi^{\epsilon_1})x_1 \otimes \dots \otimes x_m + (1 - p^{\epsilon_1})T_1\xi^{\epsilon_1} \otimes x_1 \otimes \dots \otimes x_m \\ &= \psi(E_1; T_1, \dots, T_n) + \psi(E_2; T_1, \dots, T_n), \end{aligned}$$

where $E_1, E_2 \in LR(\chi, \epsilon)$ are the diagrams constructed from D by, respectively, leaving the new top node isolated or adding a new rib and spine.

Since every $D \in LR(\chi, \epsilon)$ is constructed from exactly one diagram in $LR(\chi_0, \epsilon_0)$ we have

$$\lambda(T_1)\mu_2(T_2)\cdots\mu_n(T_n)\xi = \sum_{D \in LR(\chi, \epsilon)} \psi(D; T_1, \dots, T_n).$$

The case $\mu_1 = \rho$ is exactly the same upon replacing “leftmost” with “rightmost” and the considerations about x_1 with ones about x_m .

Now, $\varphi(\mu_1(T_1)\cdots\mu_n(T_n))$ is given by applying ψ to the left side of (1). So only the terms on the right whose vector parts are ξ will survive, that is, the terms corresponding to $E \in LR_0(\chi, \epsilon)$. Fix such a diagram and let $\sigma \in BNC(\chi, \epsilon)$ be the corresponding partition. We examine

$$\psi(E; T_1, \dots, T_n) = \prod_{W \in \sigma} \psi(W; T_1, \dots, T_n).$$

For $W = \{l_1 < \cdots < l_s\} \in \sigma$ we have

$$\begin{aligned} \psi(W; T_1, \dots, T_n) &= \psi^{\epsilon(W)} \left(T_{l_1} (1 - p^{\epsilon(W)}) T_{l_2} \cdots (1 - p^{\epsilon(V)}) T_{l_s} \xi^{\epsilon(V)} \right) \xi \\ &= \sum_{1 \leq q_1 < \cdots < q_m \leq s-1} (-1)^m \varphi^{\epsilon(W)}(T_{l_1} \cdots T_{l_{q_1}}) \cdots \varphi^{\epsilon(V)}(T_{l_{q_m+1}} \cdots T_{l_s}) \xi. \end{aligned}$$

Each term in the last sum corresponds to a lateral refinement $\pi_W = \{V_1, \dots, V_{m+1}\}$ of W , weighted by $(-1)^{|\pi_W| - |W|}$. As any lateral refinement of σ is simply a collection of lateral refinements of its individual blocks, we see that $\pi = \bigcup_{W \in \sigma} \pi_W$ is a lateral refinement of σ . The overall weight associated to π is $\prod_{W \in \sigma} (-1)^{|\pi_W| - |W|} = (-1)^{|\pi| - |\sigma|}$. Thus we obtain

$$\psi(E; T_1, \dots, T_n) = \sum_{\substack{\pi \in BNC(\chi) \\ \pi \leq_{\text{lat}} \sigma}} (-1)^{|\pi| - |\sigma|} \varphi_\pi(T_1, \dots, T_n).$$

Summing over $E \in LR_0(\chi, \epsilon)$ (or equivalently $\sigma \in BNC(\chi, \epsilon)$) and reversing the order of the two summations yields (2). \square

Corollary 4.1.2. *Let z' and z'' be a pair of two-faced families in (\mathcal{A}, φ) . Then z' and z'' are bi-free if and only if for every map $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$ and $\epsilon \in \{', ''\}^n$ we have*

$$\varphi_\alpha(z^\epsilon) = \sum_{\pi \in BNC(\alpha)} \left[\sum_{\substack{\sigma \in BNC(\alpha, \epsilon) \\ \sigma \geq_{\text{lat}} \pi}} (-1)^{|\pi| - |\sigma|} \varphi_\pi(z^\epsilon) \right] \quad (3)$$

where $z^\epsilon = (z_{\alpha(1)}^{\epsilon_1}, \dots, z_{\alpha(n)}^{\epsilon_n})$.

Proof. If z' and z'' are bi-free then this immediately follows by applying the previous proposition to the representation guaranteed by the definition of bi-freeness.

Conversely, suppose z' and z'' satisfy (3) for each α and ϵ . As in the proof of Proposition 2.9 of [5], we consider the universal representations of z' and z'' . That the joint representation in their free product is the same as the joint representation of z' and z'' follows precisely from (3). \square

4.2. Summation considerations. For $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$, $\epsilon \in \{', ''\}^n$, and $\pi \in BNC(\chi)$, we will write $\pi \leq \epsilon$ where we think of ϵ as the induced partition in $\mathcal{P}(n)$.

Proposition 4.2.1. *Let $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ and $\epsilon \in \{', ''\}^n$. Then for every $\pi \in BNC(\chi)$ such that $\pi \leq \epsilon$,*

$$\sum_{\substack{\sigma \in BNC(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi}} (-1)^{|\pi| - |\sigma|} = \sum_{\substack{\sigma \in BNC(\chi) \\ \pi \leq \sigma \leq \epsilon}} \mu_{BNC}(\pi, \sigma).$$

To prove Proposition 4.2.1 we will appeal to free probability to handle the following case and reduce all others to it.

Lemma 4.2.2. *Let $\chi : \{1, \dots, n\} \rightarrow \{\ell, r\}$ with $\chi \equiv \ell$ and $\epsilon \in \{\ell, r\}^n$. Then for every $\pi \in BNC(\chi)$ such that $\pi \leq \epsilon$,*

$$\sum_{\substack{\sigma \in BNC(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi}} (-1)^{|\pi| - |\sigma|} = \sum_{\substack{\sigma \in BNC(\chi) \\ \pi \leq \sigma \leq \epsilon}} \mu_{BNC}(\pi, \sigma).$$

Proof. Let $\{X'_1, \dots, X'_n\}$ and $\{X''_1, \dots, X''_n\}$ be freely independent sets. Note by Proposition 2.15b of [5] these sets can be viewed as a bi-free pair of two faced families X' and X'' with trivial right faces. Hence, by Corollary 4.1.2,

$$\varphi(X_1^{\epsilon_1} \dots X_n^{\epsilon_n}) = \sum_{\pi \in BNC(\chi)} \left(\sum_{\substack{\sigma \in BNC(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi}} (-1)^{|\pi| - |\sigma|} \right) \varphi_{\pi}(X_1^{\epsilon_1}, \dots, X_n^{\epsilon_n})$$

Since $\chi \equiv \ell$, $BNC(\chi) = NC(n)$. Thus, since $\{X'_1, \dots, X'_n\}$ and $\{X''_1, \dots, X''_n\}$ are free,

$$\begin{aligned} \varphi(X_1^{\epsilon_1} \dots X_n^{\epsilon_n}) &= \sum_{\sigma \in BNC(\chi)} \kappa_{\sigma}(X_1^{\epsilon_1}, \dots, X_n^{\epsilon_n}) \\ &= \sum_{\substack{\sigma \in BNC(\chi) \\ \sigma \leq \epsilon}} \kappa_{\sigma}(X_1^{\epsilon_1}, \dots, X_n^{\epsilon_n}) \\ &= \sum_{\substack{\sigma \in BNC(\chi) \\ \sigma \leq \epsilon}} \sum_{\substack{\pi \in BNC(\chi) \\ \pi \leq \sigma}} \mu(\pi, \sigma) \varphi_{\pi}(X_1^{\epsilon_1}, \dots, X_n^{\epsilon_n}) \\ &= \sum_{\substack{\pi \in BNC(\chi) \\ \pi \leq \epsilon}} \left(\sum_{\substack{\sigma \in BNC(\chi) \\ \pi \leq \sigma \leq \epsilon}} \mu(\pi, \sigma) \right) \varphi_{\pi}(X_1^{\epsilon_1}, \dots, X_n^{\epsilon_n}). \end{aligned}$$

Since these expressions agree for any selection of $\{X'_1, \dots, X'_n\}$ and $\{X''_1, \dots, X''_n\}$ that are freely independent, by selecting $\{X'_1, \dots, X'_n\}$ and $\{X''_1, \dots, X''_n\}$ that are free and such that $\varphi_{\pi}(X_1^{\epsilon_1}, \dots, X_n^{\epsilon_n})$ is non-zero for precisely one π , the desired sums are obtained to be equal (as $\mu = \mu_{BNC}$ in this setting). \square

We will use Lemma 4.2.2 to show that the desired equations in Proposition 4.2.1 hold. To do so, we will show that an arbitrary bi-non-crossing partition can be obtained by a sequence of steps, preserving the summations in Proposition 4.2.1, applied to a partition with all left nodes.

Lemma 4.2.3. *Let $\chi : \{1, \dots, n\} \rightarrow \{\ell, r\}$ with $\chi(n) = \ell$, $\epsilon \in \{\ell, r\}^n$, and $\pi \in BNC(\chi)$ be such that $\pi \leq \epsilon$. Let $\hat{\chi} : \{1, \dots, n\} \rightarrow \{\ell, r\}$ be such that*

$$\hat{\chi}(t) = \begin{cases} \chi(t) & \text{if } t \neq n \\ r & \text{if } t = n \end{cases},$$

and let $\hat{\pi} \in BNC(\hat{\chi})$ be the unique shaded bi-non-crossing partition with the same blocks as π (note $\hat{\pi} \leq \epsilon$ by construction). Then

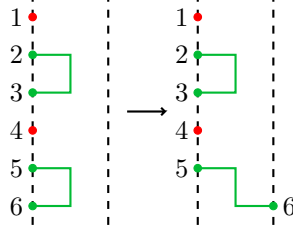
$$\sum_{\substack{\sigma \in BNC(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi}} (-1)^{|\pi| - |\sigma|} = \sum_{\substack{\hat{\sigma} \in BNC(\hat{\chi}, \epsilon) \\ \hat{\sigma} \geq_{\text{lat}} \hat{\pi}}} (-1)^{|\hat{\pi}| - |\hat{\sigma}|}$$

and

$$\sum_{\substack{\sigma \in BNC(\chi) \\ \pi \leq \sigma \leq \epsilon}} \mu_{BNC}(\pi, \sigma) = \sum_{\substack{\hat{\sigma} \in BNC(\hat{\chi}) \\ \hat{\pi} \leq \hat{\sigma} \leq \epsilon}} \mu_{BNC}(\hat{\pi}, \hat{\sigma}).$$

Proof. It is clear that the operator which takes an element $\sigma \in BNC(\chi, \epsilon)$ and constructs an element $\hat{\sigma} \in BNC(\hat{\chi}, \epsilon)$ with the same blocks as σ corresponds to taking the bottom node of σ which is on the

left and placing this node on the right (keeping all strings connected). For example, consider the following diagrams.



Such operation is clearly a bijection, maps $BNC(\chi, \epsilon)$ to $BNC(\hat{\chi}, \epsilon)$, $(-1)^{|\pi| - |\sigma|} = (-1)^{|\hat{\pi}| - |\hat{\sigma}|}$, and $\sigma \geq_{\text{lat}} \pi$ if and only if $\hat{\sigma} \geq_{\text{lat}} \hat{\pi}$. Hence the first equation holds. Similarly, by Remarks 3.1.4, it is clear that the second equation holds. \square

Lemma 4.2.4. *Let $\chi : \{1, \dots, n\} \rightarrow \{\ell, r\}$ be such that there exists a $k \in \{1, \dots, n-1\}$ such that $\chi(k) = \ell$ and $\chi(k+1) = r$, $\epsilon \in \{', ''\}^n$, and $\pi \in BNC(\chi)$ be such that $\pi \leq \epsilon$. Fix $k \in \{1, \dots, n-1\}$. Let $\hat{\epsilon} \in \{', ''\}^n$ be such that*

$$\hat{\epsilon}_t = \begin{cases} \epsilon_t & \text{if } t \notin \{k, k+1\} \\ \epsilon_k & \text{if } t = k+1 \\ \epsilon_{k+1} & \text{if } t = k \end{cases},$$

let $\hat{\chi} : \{1, \dots, n\} \rightarrow \{\ell, r\}$ be such that

$$\hat{\chi}(t) = \begin{cases} \chi(t) & \text{if } t \notin \{k, k+1\} \\ \chi(k) & \text{if } t = k+1 \\ \chi(k+1) & \text{if } t = k \end{cases},$$

and let $\hat{\pi} \in BNC(\hat{\chi})$ be the unique shaded bi-non-crossing partition obtained by interchanging k and $k+1$ in π (note $\hat{\pi} \leq \hat{\epsilon}$ by construction). Then

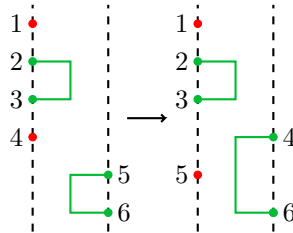
$$\sum_{\substack{\sigma \in BNC(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi}} (-1)^{|\pi| - |\sigma|} = \sum_{\substack{\hat{\sigma} \in BNC(\hat{\chi}, \hat{\epsilon}) \\ \hat{\sigma} \geq_{\text{lat}} \hat{\pi}}} (-1)^{|\hat{\pi}| - |\hat{\sigma}|}$$

and

$$\sum_{\substack{\sigma \in BNC(\chi) \\ \pi \leq \sigma \leq \epsilon}} \mu_{BNC}(\pi, \sigma) = \sum_{\substack{\hat{\sigma} \in BNC(\hat{\chi}) \\ \hat{\pi} \leq \hat{\sigma} \leq \hat{\epsilon}}} \mu_{BNC}(\hat{\pi}, \hat{\sigma}).$$

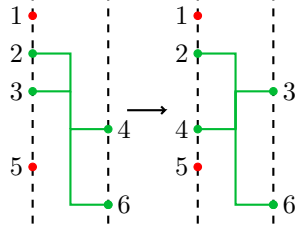
Proof. Since the operation that takes an element $\sigma \in BNC(\chi)$ with $\sigma \leq \epsilon$ and produces an element $\hat{\sigma} \in BNC(\hat{\chi})$ with $\hat{\sigma} \leq \hat{\epsilon}$ by interchanging k and $k+1$ in σ is a bijection, and since $\mu_{BNC}(\pi, \sigma) = \mu_{BNC}(\hat{\pi}, \hat{\sigma})$ by Remarks 3.1.4, the second equation clearly holds.

To prove the first equation holds, we break the discussion into several cases. For the first case, suppose $\epsilon_k \neq \epsilon_{k+1}$; that is, the nodes we desired to change the orders of are of different shades. For example, see the following diagrams where $k = 4$.



In this case it is clear that the operation that takes $\sigma \in BNC(\chi)$ to $\hat{\sigma} \in BNC(\hat{\chi})$ described above is a bijection that maps $BNC(\chi, \epsilon)$ to $BNC(\hat{\chi}, \hat{\epsilon})$, is such that $(-1)^{|\pi| - |\sigma|} = (-1)^{|\hat{\pi}| - |\hat{\sigma}|}$, and is such that $\sigma \geq_{\text{lat}} \pi$ if and only if $\hat{\sigma} \geq_{\text{lat}} \hat{\pi}$. Hence the first equation holds in this case.

Otherwise $\epsilon_k = \epsilon_{k+1}$. Suppose k and $k+1$ are in the same block of π . For example, consider the following diagrams where $k = 3$.



It is again clear that the same identifications as the previous case hold and thus the first equation holds in this case. Hence we have reduced to the case that k and $k+1$ are in different blocks of the same shade.

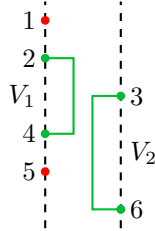
Let V_1 and V_2 be the blocks in π of k and $k+1$ respectively. Note that V_1 contains a left node and V_2 contains a right node and the sum on the left-hand-side of the first equation is

$$\sum_{\substack{\sigma \in BNC(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi \\ k, k+1 \text{ in separated blocks of } \sigma}} (-1)^{|\pi| - |\sigma|} + \sum_{\substack{\sigma \in BNC(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi \\ k, k+1 \text{ not in separated blocks of } \sigma}} (-1)^{|\pi| - |\sigma|}.$$

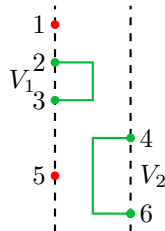
We claim that

$$\sum_{\substack{\sigma \in BNC(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi \\ k, k+1 \text{ not in separated blocks of } \sigma}} (-1)^{|\pi| - |\sigma|} = 0.$$

Indeed we will split the discussion into two cases: when V_1 and V_2 are piled and when they are not. For an example when V_1 and V_2 are piled, consider the following diagram.



If V_1 and V_2 are piled, it is easy to see that any $\sigma \in BNC(\chi, \epsilon)$ such that $\pi \leq \sigma$ and k and $k+1$ are not in separated blocks of σ must be such that V_1 and V_2 are contained in the same block of σ . However, this implies that π is not a lateral refinement of σ as joining piled blocks cannot be undone by a lateral refinement. Hence the sum is zero in this case. Otherwise, suppose V_1 and V_2 are not piled. For an example where V_1 and V_2 are not piled, consider the following diagram.



This implies k is the lowest element of V_1 in the bi-non-crossing diagram of π and $k+1$ is the highest element of V_2 . If $\sigma \in BNC(\chi, \epsilon)$ is such that k and $k+1$ are not in separated blocks of σ and $\sigma \geq \pi$, then if k and $k+1$ are in the same block of σ , let $\sigma' \leq_{\text{lat}} \sigma$ splitting the block containing k and $k+1$ inbetween these node (note $\sigma' \in BNC(\chi, \epsilon)$). Otherwise k and $k+1$ are not in the same block of σ so letting $\sigma' \geq_{\text{lat}} \sigma$ be the partition made by joining the blocks containing k and $k+1$ together also forms a partition in $BNC(\chi, \epsilon)$. In either case $(-1)^{|\pi| - |\sigma|} + (-1)^{|\pi| - |\sigma'|} = 0$. Note that the correspondance between σ and σ' in each case is one-to-one and thus the sum is zero.

Similar arguments show that

$$\sum_{\substack{\hat{\sigma} \in BNC(\hat{\chi}, \hat{\epsilon}) \\ \hat{\sigma} \geq_{\text{lat}} \hat{\pi}}} (-1)^{|\hat{\pi}| - |\hat{\sigma}|} = \sum_{\substack{\hat{\sigma} \in BNC(\hat{\chi}, \hat{\epsilon}) \\ \hat{\sigma} \geq_{\text{lat}} \hat{\pi} \\ k, k+1 \text{ in separated blocks of } \hat{\sigma}}} (-1)^{|\hat{\pi}| - |\hat{\sigma}|}.$$

However, the map taking $\sigma \in BNC(\chi)$ to $\hat{\sigma} \in BNC(\hat{\chi})$ is such that k and $k+1$ are in separated blocks of σ if and only if k and $k+1$ are in separated blocks of $\hat{\sigma}$, and under these conditions $\sigma \in BNC(\chi, \epsilon)$ if and only if $\hat{\sigma} \in BNC(\hat{\chi}, \hat{\epsilon})$, $\sigma \geq_{\text{lat}} \pi$ if and only if $\hat{\sigma} \geq_{\text{lat}} \hat{\pi}$, and $(-1)^{|\pi| - |\sigma|} = (-1)^{|\hat{\pi}| - |\hat{\sigma}|}$. Hence the first equation holds in this final case. \square

Proof of Proposition 4.2.1. Given π , a $\hat{\pi}$ in $BNC(\hat{\chi})$ where $\hat{\chi} : \{1, \dots, n\} \rightarrow \{\ell\}$ may be constructed such that $\hat{\pi}$ can be modified to make π via the operations in used in Lemma 4.2.3 and Lemma 4.2.4. Since the sums are equal for $\hat{\pi}$ by Lemma 4.2.2 and since Lemma 4.2.3 and Lemma 4.2.4 preserve the equality of the sums, the result hold for π . \square

We apply Proposition 4.2.1 to Corollary 4.1.2 to immediately obtain the following.

Corollary 4.2.5. *Let z' and z'' be a pair of two-faced families in (\mathcal{A}, φ) . Then z' and z'' are bi-free if and only if for every map $\alpha : \{1, \dots, n\} \rightarrow I \sqcup J$ and $\epsilon \in \{', ''\}^n$ we have*

$$\varphi_{\alpha}(z^{\epsilon}) = \sum_{\pi \in BNC(\alpha)} \left[\sum_{\substack{\sigma \in BNC(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{BNC}(\pi, \sigma) \right] \varphi_{\pi}(z^{\epsilon}), \quad (4)$$

where $z^{\epsilon} = (z_{\alpha(1)}^{\epsilon_1}, \dots, z_{\alpha(n)}^{\epsilon_n})$.

4.3. Bi-free is equivalent to combinatorially-bi-free.

Theorem 4.3.1. *Let $z' = ((z'_i)_{i \in I}, (z'_j)_{j \in J})$ and $z'' = ((z''_i)_{i \in I}, (z''_j)_{j \in J})$ be a pair of two-faced families in a non-commutative probability space (\mathcal{A}, φ) . Then z' and z'' are bi-free if and only if they are combinatorially-bi-free.*

Proof. Suppose z' and z'' are bi-free, and fix a shading $\epsilon \in \{', ''\}^n$. By Corollary 4.2.5, for $\alpha : \{1, \dots, n\} \rightarrow I \sqcup J$ we have

$$\varphi_{\alpha}(z^{\epsilon}) = \sum_{\pi \in BNC(\alpha)} \left(\sum_{\substack{\sigma \in BNC(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{BNC}(\pi, \sigma) \right) \varphi_{\pi}(z^{\epsilon}).$$

Therefore

$$\varphi_{\alpha}(z^{\epsilon}) = \sum_{\substack{\sigma \in BNC(\alpha) \\ \sigma \leq \epsilon}} \kappa_{\sigma}(z^{\epsilon})$$

by Remark 3.1.5. Using the above formula, we will proceed inductively to show that $\kappa_{\sigma}(z^{\epsilon}) = 0$ if $\sigma \in BNC(\alpha)$ and $\sigma \not\leq \epsilon$. The base case is where $n = 1$ is immediate.

For the inductive case, suppose the result holds for any $\beta : \{1, \dots, k\} \rightarrow I \sqcup J$ with $k < n$. Let $\alpha : \{1, \dots, n\} \rightarrow I \sqcup J$. Suppose ϵ is not constant (so in particular, $1_{\alpha} \not\leq \epsilon$). Then

$$\sum_{\sigma \in BNC(\alpha)} \kappa_{\sigma}(z^{\epsilon}) = \varphi_{\alpha}(z^{\epsilon}) = \sum_{\substack{\sigma \in BNC(\alpha) \\ \sigma \leq \epsilon}} \kappa_{\sigma}(z^{\epsilon}).$$

By induction, $\kappa_{\sigma}(z^{\epsilon}) = 0$ if $\sigma \in BNC(\alpha) \setminus \{1_{\alpha}\}$ and $\sigma \not\leq \epsilon$. Consequently

$$\sum_{\sigma \in BNC(\alpha)} \kappa_{\sigma}(z^{\epsilon}) = \kappa_{1_{\alpha}}(z^{\epsilon}) + \sum_{\substack{\sigma \in BNC(\alpha) \\ \sigma \leq \epsilon}} \kappa_{\sigma}(z^{\epsilon}).$$

Combining these two equations gives $\kappa_{1_{\alpha}}(z^{\epsilon}) = 0$ completing the inductive step.

Now suppose z' and z'' are combinatorially-bi-free. Then, for any $\alpha : \{1, \dots, n\} \rightarrow I \sqcup J$ and $\epsilon \in \{', ''\}^n$,

$$\begin{aligned} \varphi_\alpha(z^\epsilon) &= \sum_{\sigma \in \text{BNC}(\alpha)} \kappa_\sigma(z^\epsilon) = \sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \sigma \leq \epsilon}} \kappa_\sigma(z^\epsilon) \\ &= \sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \sigma \leq \epsilon}} \sum_{\substack{\pi \in \text{BNC}(\alpha) \\ \pi \leq \sigma}} \varphi_\pi(z^\epsilon) \mu_{\text{BNC}}(\pi, \sigma) \\ &= \sum_{\pi \in \text{BNC}(\alpha)} \left(\sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{\text{BNC}}(\pi, \sigma) \right) \varphi_\pi(z^\epsilon). \end{aligned}$$

Hence Corollary 4.2.5 implies that z' and z'' are bi-free. \square

4.4. Voiculescu's universal bi-free polynomials. Using the equivalence of bi-free independence and combinatorial-bi-free independence we obtain explicit formulas for several universal polynomials appearing [5].

Proposition 4.4.1. *Let $\alpha : \{1, \dots, n\} \rightarrow I \sqcup J$. For each shading $\epsilon \in \{', ''\}^n$ we define a polynomial $P_{\alpha, \epsilon}$ on indeterminates X'_K and X''_K indexed by non-empty subsets $K \subset \{1, \dots, n\}$ by the formula*

$$P_{\alpha, \epsilon} := \sum_{\pi \in \text{BNC}(\alpha, \epsilon)} \left[\sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{\text{BNC}}(\pi, \sigma) \right] \prod_{V \in \pi} X_V^{\epsilon(V)}.$$

Then for z' and z'' a bi-free pair of two-faced families in (\mathcal{A}, φ) we have

$$\varphi_\alpha(z^\epsilon) = P_{\alpha, \epsilon}(z', z''),$$

where $P_{\alpha, \epsilon}(z', z'')$ is given by evaluating $P_{\alpha, \epsilon}$ at $X_{\{k_1 < \dots < k_r\}}^\delta = \varphi(z_{\alpha(k_1)}^\delta \cdots z_{\alpha(k_r)}^\delta)$, $\delta \in \{', ''\}$.

Furthermore, if we define Q_α as the sum of the $P_{\alpha, \epsilon}$ over all possible shadings then

$$Q_\alpha = X'_{\{1, \dots, n\}} + X''_{\{1, \dots, n\}} + \sum P_{\alpha, \epsilon},$$

where the summation is over non-constant shadings ϵ , and

$$\varphi_\alpha(z' + z'') = Q_\alpha(z', z''),$$

where $Q_\alpha(z', z'')$ is Q_α evaluated at the same point as the $P_{\alpha, \epsilon}$ above.

Proof. The first part of this corollary is immediate from Corollary 4.2.5. The assertion regarding $Q_\alpha(z', z'')$ is also immediate when expanding the product in the left-hand side. All that remains to show is

$$Q_\alpha = X'_{\{1, \dots, n\}} + X''_{\{1, \dots, n\}} + \sum P_{\alpha, \epsilon},$$

which is equivalent to saying $P_{\alpha, \epsilon} = X_{\{1, \dots, n\}}^\delta$ when ϵ is the constant shading $\epsilon = (\delta, \dots, \delta)$, $\delta \in \{', ''\}$. Such a shading induces the full partition 1_α , and hence

$$\sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{\text{BNC}}(\pi, \sigma) = \sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \pi \leq \sigma \leq 1_\alpha}} \mu_{\text{BNC}}(\pi, \sigma) = \delta_{\text{BNC}}(\pi, 1_\alpha).$$

Then the only term in $P_{\alpha, \epsilon}$ with a non-zero coefficient is the one corresponding to $\pi = 1_\alpha$. \square

Proposition 4.4.2. *For any $\alpha : \{1, \dots, n\} \rightarrow I \sqcup J$, recursively define polynomials R_α on indeterminates X_K indexed by non-empty subsets $K \subseteq \{1, \dots, n\}$ by the formula*

$$R_\alpha = \sum_{\pi \in \text{BNC}(\alpha)} \mu_{\text{BNC}}(\pi, 1_\alpha) \prod_{V \in \pi} X_V$$

If X_K is given degree $|K|$, then R_α is homogeneous with degree n .

For z a two-faced family in (\mathcal{A}, φ) , if $R_\alpha(z)$ denotes R_α evaluated at the point $X_{\{k_1 < \dots < k_r\}} = \varphi(z_{\alpha(k_1)} \cdots z_{\alpha(k_r)})$ then $R_\alpha(z) = \kappa_\alpha(z)$. Moreover, if z' and z'' are bi-free in (\mathcal{A}, φ) then $R_\alpha(z' + z'') = R_\alpha(z') + R_\alpha(z'')$; that is, R_α has the cumulant property.

Proof. We see that $R_\alpha(z)$ and $\kappa_\alpha(z)$ are equal by Remark 3.1.5). Then R_α has the cumulant property simply because κ_α does. \square

Remark 4.4.3. The polynomials $P_{\alpha,\epsilon}$, Q_α , and R_α are precisely the universal polynomials from Propositions 2.18, 5.6, and Theorem 5.7, respectively, in [5].

5. A MULTIPLICATIVE BI-FREE CONVOLUTION

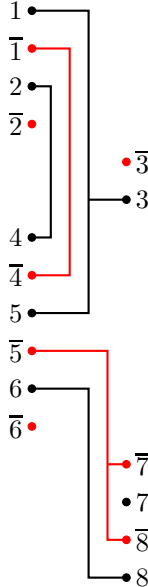
5.1. Kreweras complement on bi-non-crossing partitions. In [3], the Kreweras complement K_{NC} on the non-crossing partitions was used to simplify the convolution of multiplicative functions. In particular, we have the following extension to BNC .

Definition 5.1.1. For any $\chi : \{1, \dots, n\} \rightarrow I \sqcup J$ and $\pi \in BNC(\chi)$, the *Kreweras complement* of π , denoted $K_{BNC}(\pi)$, is the element of $BNC(\chi)$ obtained by applying s_χ to the Kreweras complement in $NC(n)$ of $s_\chi^{-1} \cdot \pi$; explicitly

$$K_{BNC}(\pi) = s_\chi \cdot K_{NC}(s_\chi^{-1} \cdot \pi).$$

Remark 5.1.2. Note that $K_{BNC}(\pi)$ may be obtained by taking the diagram corresponding to π , placing a node beneath each left node and above each right node of π , and drawing the largest bi-non-crossing diagram on the new nodes.

Example 5.1.3. In the following diagram, if π is the bi-non-crossing partition drawn in black, $K_{BNC}(\pi)$ is the bi-non-crossing partition in red.



Remark 5.1.4. Since K_{NC} is an order reversing and s_χ is order preserving, K_{BNC} is an order reversing bijection. Thus $[\pi, 1_\alpha] \simeq [K_{BNC}(1_\alpha), K_{BNC}(\pi)] = [0_\alpha, K_{BNC}(\pi)]$ for all $\pi \in BNC(\alpha)$. Hence, if $f, g \in IA(BNC)$ are multiplicative functions, then

$$(f * g)(0_\alpha, 1_\alpha) = \sum_{\pi \in BNC(\alpha)} f(0_\alpha, \pi) g(0_\alpha, K_{BNC}(\pi)) = (g * f)(0_\alpha, 1_\alpha)$$

and thus $f * g = g * f$.

5.2. Computing cumulants of a multiplicative bi-free convolution. Taking inspiration from [3], we use the Kreweras complement to examine the bi-free cumulants of a two-faced family generated by products of a bi-free pair of two-faced families.

Theorem 5.2.1. Let $z' = (\{z'_\ell\}, \{z'_r\})$ and $z'' = (\{z''_\ell\}, \{z''_r\})$ be a bi-free family of pairs of faces and let $z = (\{z'_\ell z''_\ell\}, \{z'_r z''_r\})$. Then

$$\kappa_\chi(z) = \sum_{\pi \in BNC(\chi)} \kappa_\pi(z') \kappa_{K_{BNC}(\chi)}(z'')$$

for all $\chi : \{1, \dots, n\} \rightarrow \{\ell, r\}$.

Proof. Since the bi-free cumulant functions are multiplicative and by the structure of the convolution of multiplicative functions given in Remark 5.1.4, it suffices to show $\kappa_z = \kappa_{z'} * \kappa_{z''}$. Using the relations $m_z * \mu_{BNC} = \kappa_z$ and $\kappa_z * \zeta_{BNC} = m_z$, it suffices to show $m_z = \kappa_{z'} * m_{z''}$.

Suppose $\chi : \{1, \dots, n\} \rightarrow \{\ell, r\}$. Let $\beta : \{1, \dots, 2n\} \rightarrow \{\ell, r\}$ be given by $\beta(2k-1) = \beta_{2k} = \chi(k)$. Take $\epsilon \in \{', ''\}^{2n}$ so that $\epsilon_{2k-1} = '$ and $\epsilon_{2k} = ''$ if $k \in \chi^{-1}(\ell)$, and the opposite if $k \in \chi^{-1}(r)$. Then

$$\begin{aligned} \varphi_\chi(z) &= \varphi\left(z_{\chi(1)}^{\epsilon_1} z_{\chi(1)}^{\epsilon_2} \cdots z_{\chi(n)}^{\epsilon_{2n-1}} z_{\chi(n)}^{\epsilon_{2n}}\right) \\ &= \varphi\left(z_{\beta(1)}^{\epsilon_1} z_{\beta(2)}^{\epsilon_2} \cdots z_{\beta(2n-1)}^{\epsilon_{2n-1}} z_{\beta(2n)}^{\epsilon_{2n}}\right) \\ &= \sum_{\pi \in BNC(\beta, \epsilon)} \kappa_\pi(z^\epsilon) \\ &= \sum_{\pi_1 \in BNC(\chi)} \kappa_{\pi_1}(z') \sum_{\substack{\pi_2 \in BNC(\chi) \\ \pi_2 \leq K_{BNC}(\pi_1)}} \kappa_{\pi_2}(z'') \\ &= \sum_{\pi_1 \in BNC(\chi)} \kappa_{\pi_1}(z') \varphi_{K_{BNC}(\pi_1)}(z'') \\ &= (\kappa_{z'} * m_{z''})(0_\chi, 1_\chi) \end{aligned}$$

Hence, as m_z and $\kappa_{z'} * m_{z''}$ are multiplicative functions that agree on full lattices in BNC , the result follows. \square

Remark 5.2.2. Note that the above generalizes the formula for the free cumulants of the multiplicative convolution of freely independent random variables in terms of their individual cumulants (*cf.* Section 3.5 of [3]). This seems to suggest that when defining the multiplicative convolution of a bi-free pair of two-faced families one should multiply the right faces as if in the opposite algebra.

Remark 5.2.3. Since convolution is abelian on multiplicative functions, we obtain that $(\{z'_\ell z''_\ell\}, \{z''_r z'_r\})$ and $(\{z'_\ell z'_\ell\}, \{z''_r z''_r\})$ have the same joint distributions.

6. AN OPERATOR MODEL FOR PAIRS OF FACES

In this section we will construct an operator model for a two-faced family in a non-commutative probability space. This model will generalize the operator model usually considered in free probability introduced by Nica in [2].

In Definition 3.2.1 of [2] Nica's operator model is constructed via unbounded operators on a Fock space making use of the left creation and annihilation operators where each product of creation operators is weighted by a free cumulant of the random variables. The operator model for a pair of faces in a non-commutative probability space will be constructed in Theorem 6.4.1 as operators on a Fock space using left annihilation operators plus special operators weighted by the corresponding bi-free cumulants. These special operators act on certain vectors differently but ultimately behave like creation operators where the creation may occur in multiple places on the left, right, or in the middle of a tensor of the Fock space. We should point out that our model does reduce directly to Nica's model in the case all of our variables are left (or right) variables and that a model using only left and right creation and annihilation operators is unlikely by discussions in [1].

Nica's operator model also gives a direct analogue to the R -series of a collection of random variables in a non-commutative probability space. Thus the operator Θ_z in Theorem 6.4.1 really is the non-commutative R -series of the two-faced family $z = (\{z_i\}_{i \in I}, \{z_j\}_{j \in J})$. In particular, if

$$z' = (\{z'_i\}_{i \in I}, \{z'_j\}_{j \in J}) \text{ and } z'' = (\{z''_i\}_{i \in I}, \{z''_j\}_{j \in J})$$

is a bi-free pair of two-faced families, we can consider the single family

$$z = (\{z'_i, z''_i\}_{i \in I}, \{z'_j, z''_j\}_{j \in J})$$

and construct the corresponding operator Θ_z . It will follow that

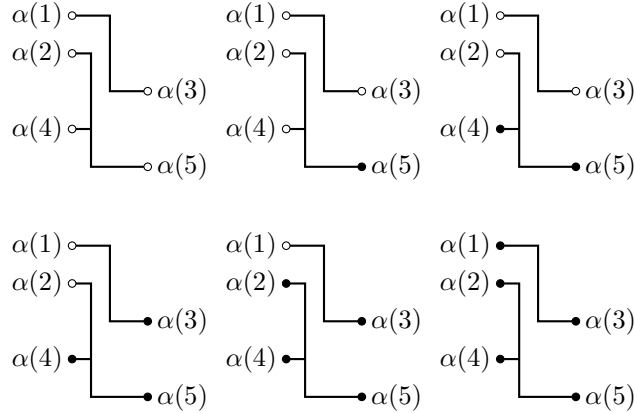
$$\Theta_z - I = (\Theta_{z'} - I) + (\Theta_{z''} - I)$$

(where $\Theta_{z'}$ and $\Theta_{z''}$ have been extended to the potentially larger Fock space on which Θ_z acts). Hence the operator Θ_z from Theorem 6.4.1 does indeed behave like an R -series.

6.1. Skeletons corresponding to bi-non-crossing partitions. The operator model from [2] can be thought of as a systematic way of constructing all non-crossing partitions weighted by products of free cumulants. The main tool in explaining the operator model for a two-faced family is to understand how one can construct bi-non-crossing partitions sequentially.

Definition 6.1.1. Let $\alpha : \{1, \dots, n\} \rightarrow I \sqcup J$. For a bi-non-crossing partition $\pi \in BNC(\alpha)$, the *skeletons* on π are the diagrams of Subsection 2.4 with the following conventions: the k^{th} -node from the top is labelled $\alpha(k)$; each node is either an open circle or a closed circle; and if node $\alpha(k)$ is open, then node $\alpha(k')$ is open for all $k' < k$.

Example 6.1.2. If α and π are as in Example 2.4.1, the skeletons corresponding to π are the following diagrams.



Definition 6.1.3. We will refer to a skeleton where all nodes are closed circles as the *completed skeleton*. For a skeleton on $1_\alpha \in BNC(\alpha)$, the skeleton where all nodes are open will be referred to as the *empty skeleton* corresponding to α , while the skeleton where all but the bottom node is open will be referred to as the *starter skeleton* corresponding to α . Any skeleton that is not empty will be referred to as a *partially completed skeleton*.

6.2. Nica's model explained via skeletons. In this subsection, we will outline how Nica's operator model from [2] can be expressed with skeletons.

Construction 6.2.1. Let $X = \{X_i\}_{i \in I}$ be a collection of random variables in a non-commutative probability space (\mathcal{A}, φ) . Let $\mathcal{F}(\mathbb{C}^{|I|})$ be the Fock space with $|I|$ generators; that is,

$$\mathcal{F}(\mathbb{C}^{|I|}) := \mathbb{C}\Omega \oplus \left(\bigoplus_{\substack{k \geq 1 \\ i_1, \dots, i_k \in I}} \mathbb{C}(e_{i_1} \otimes \dots \otimes e_{i_k}) \right)$$

where $\{e_i\}_{i \in I}$ is a fixed orthonormal basis of $\mathbb{C}^{|I|}$. The vector Ω is called the vacuum vector of $\mathcal{F}(\mathbb{C}^{|I|})$.

For $i \in I$, the creation operator corresponding to e_i , denoted L_i , is the unique linear operator such that $L_i(\Omega) = e_i$ and

$$L_i(e_{i_1} \otimes \dots \otimes e_{i_k}) = e_i \otimes e_{i_1} \otimes \dots \otimes e_{i_k}$$

whenever $k \geq 1$ and $i_1, \dots, i_k \in I$. The annihilation operator corresponding to e_i is L_i^* .

Let $\omega : \mathcal{L}(\mathcal{F}(\mathbb{C}^{|I|})) \rightarrow \mathbb{C}$ be defined by $\omega(T) = \langle T\Omega, \Omega \rangle$ and consider the (unbounded) operator

$$\Theta_X := I_{\mathcal{F}(\mathbb{C}^{|I|})} + \sum_{k \geq 1} \sum_{i_1, \dots, i_k \in I} \kappa_{NC}(X_{i_1}, \dots, X_{i_k}) L_{i_k} \cdots L_{i_1}$$

where $\kappa_{NC}(X_{i_1}, \dots, X_{i_k})$ is the free cumulant of the tuple $(X_{i_1}, \dots, X_{i_k})$ (cf. [4]).

The joint distribution of the operators

$$Z_i := L_i^* \Theta = L_i^* + \sum_{k \geq 0} \sum_{i_1, \dots, i_k \in I} \kappa(X_{i_1}, \dots, X_{i_k}, X_i) L_{i_k} \cdots L_{i_1}$$

with respect to ω is the same as the joint distribution of $\{X_i\}_{i \in I}$ with respect to φ .

Remark 6.2.2. One way to show that the joint distribution $\{Z_i\}_{i \in I}$ with respect to ω is the same as the joint distribution of $\{X_i\}_{i \in I}$ with respect to φ is as follows. Fix a sequence $Z_{i'_1} \cdots Z_{i'_n}$ for some $n \geq 1$ and $i'_1, \dots, i'_n \in I$, and consider each product $T_{i'_1,1} \cdots T_{i'_n,n}$ where

$$T_{i'_t,t} \in \{L_{i'_t}^*\} \cup \{\kappa(X_{i_1}, \dots, X_{i_k}, X_{i'_t}) L_{i_k} \cdots L_{i_1} \mid k \geq 0, i_1, \dots, i_k \in I\}.$$

Then show that there is a bijection between all non-crossing partitions in $NC(n)$ and all products $T_{i'_1,1} \cdots T_{i'_n,n}$ such that

$$\langle T_{i'_1,1} \cdots T_{i'_n,n} \Omega, \Omega \rangle \neq 0.$$

One then demonstrates that the above inner products are the correct free cumulants so that summing over $NC(n)$ yields

$$\sum_{\pi \in NC(n)} \kappa_{\pi}(X_{i_1}, \dots, X_{i_n}) = \varphi(X_{i_1} \cdots X_{i_n}).$$

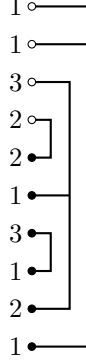
We will demonstrate a proof of the above result in terms of skeletons. We think of the free case a sub-case of the bi-free case, where all variables come from the left face. Using the notation from Remark 6.2.2, one can think of $T_{i'_1,1} \cdots T_{i'_n,n} \Omega$ as a partially completed skeleton weighted by a scalar (which is a product of free cumulants). There is not a bijection between partially completed skeletons and basis vectors of our Fock space as the partially completed skeleton will retain the information of how the vector was created. Each $L_{i'_t}^*$ acts on the skeleton by filling in the lowest open node if it is labelled i'_t (to make the node closed in the new skeleton), and otherwise weights the skeleton by zero (which removes the skeleton from consideration). For example,

$$\begin{array}{c} \begin{array}{|c} \hline 2 \circ \\ 1 \circ \\ 3 \circ \\ 2 \bullet \\ 1 \bullet \\ \hline \end{array} \xrightarrow{L_2^*} 0 \end{array} \quad \text{whereas} \quad \begin{array}{c} \begin{array}{|c} \hline 2 \circ \\ 1 \circ \\ 3 \circ \\ 2 \bullet \\ 1 \bullet \\ \hline \end{array} \xrightarrow{L_3^*} \begin{array}{|c} \hline 2 \circ \\ 1 \circ \\ 3 \bullet \\ 2 \bullet \\ 1 \bullet \\ \hline \end{array} \end{array}.$$

Each $\kappa(X_{i_1}, \dots, X_{i_k}, X_{i'_t}) L_{i'_t}^* L_{i'_t} L_{i_k} \cdots L_{i_1}$ will multiply the weight of the skeleton by $\kappa(X_{i_1}, \dots, X_{i_k}, X_{i'_t})$ and insert the starter skeleton corresponding to $\alpha : \{1, \dots, k+1\} \rightarrow I$ where $\alpha(t) = i_t$ if $t \leq k$ and $\alpha(k+1) = i'_t$ in the only possible position (that is, directly above the last completed node or at the bottom for the empty skeleton). Alternatively, one can view $L_{i'_t}^* L_{i_k} \cdots L_{i_1}$ as adding the empty skeleton corresponding to α and then $L_{i'_t}^*$ is immediately applied to fill in the lowest open node. For example,

$$\begin{array}{c} \begin{array}{|c} \hline 2 \circ \\ 1 \circ \\ 3 \circ \\ 2 \bullet \\ 1 \bullet \\ \hline \end{array} \xrightarrow{L_1^* L_1 L_2 L_1 L_3} \begin{array}{|c} \hline 2 \circ \\ 1 \circ \\ 3 \circ \\ 1 \circ \\ 2 \circ \\ 1 \bullet \\ 2 \bullet \\ 1 \bullet \\ \hline \end{array} \end{array}$$

For a product $T_{i'_1,1} \cdots T_{i'_n,n} \Omega$, we will get precisely one partially completed skeleton. For example,



is constructed by first applying $L_1^* L_1 L_1 L_1$ (weighted by $\kappa(X_1, X_1, X_1)$), then $L_2^* L_2 L_1 L_3$ (weighted by $\kappa(X_3, X_1, X_2)$), then $L_1^* L_1 L_3$ (weighted by $\kappa(X_3, X_1)$), then L_3^* , then L_1^* , and then $L_2^* L_2 L_2$ (weighted by $\kappa(X_2, X_2)$). Notice that if the above operators are applied to Ω in the order listed, we obtain the vector

$$e_2 \otimes e_3 \otimes e_1 \otimes e_1.$$

The indices of the tensor can be seen in the partially completed skeleton by reading the open nodes from bottom to top. Thus the vector $T_{i'_1,1} \cdots T_{i'_n,n} \Omega$ corresponds to a partially completed skeleton and the only products such that

$$\langle T_{i'_1,1} \cdots T_{i'_n,n} \Omega, \Omega \rangle \neq 0$$

correspond to completed skeletons. It is easy to see that a completed skeleton corresponds to an element of $\pi \in NC(n)$ with the vertices labeled by reading the skeleton from top to bottom and placing the labels left to right. These completed skeletons are weighted by the correct product of cumulants so that when we sum over all completed skeletons, we get

$$\sum_{\pi \in NC(n)} \kappa_{\pi}(X_{i_1}, \dots, X_{i_n}) = \varphi(X_{i_1} \cdots X_{i_n}),$$

as desired.

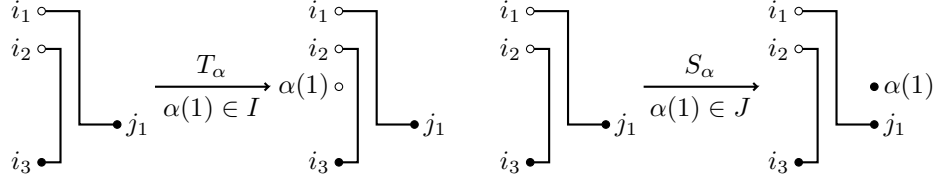
6.3. A sub-construction of the operator model. We will now begin the construction of our operator model for pairs of faces. The idea is to define operators on a Fock space so that when a product of a sequence of such operators is taken, vectors corresponding to partially completed skeletons of bi-non-crossing partitions are obtained. The main difficulty and difference from Nica's operator model is that bi-non-crossing partitions may have nodes on both the left and right, which provides both restrictions and options for how a starter skeleton may be added to a partially completed skeleton.

Construction 6.3.1. Let $z = ((z_i)_{i \in I}, (z_j)_{j \in J})$ be a two-faced family in (\mathcal{A}, φ) . Consider the Fock space $\mathcal{H} := \mathcal{F}(\mathbb{C}^{|I|+|J|})$ with $\{e_k\}_{k \in I \sqcup J}$ an orthonormal basis for $\mathbb{C}^{|I|+|J|}$ and let L_k be the left creation operator corresponding to e_k . To simplify notation in what follows, we will think of the empty tensor as Ω (e.g., $e_1 \otimes \cdots \otimes e_{k-1}$ should be thought of as Ω if $k = 1$).

For $\alpha : \{1, \dots, n\} \rightarrow I \sqcup J$, we will define operators $S_{\alpha}, T_{\alpha} \in \mathcal{L}(\mathcal{H})$. We will construct these operators on \mathcal{H} but one should think of S_{α} and T_{α} as operations on partially completed skeletons where a partially completed skeleton corresponds to a tensor of basis vectors by reading the open nodes from the bottom up and placing the corresponding element of $\{e_k\}_{k \in I \sqcup J}$ in the tensor from left to right. One should think of S_{α} as an operator that adds a starter skeleton in all possible ways and L_k^* as the operator which closes the lowest node if that node is labelled k and otherwise will multiply the weight of a skeleton by zero. In our definitions, we will always have the relation $L_{\alpha(n)} S_{\alpha} = T_{\alpha}$, so $L_k^* T_{\alpha} = \delta_{k, \alpha(n)} S_{\alpha}$.

For $n = 1$ and $\alpha(1) = k \in I \sqcup J$, we define $T_{\alpha} := L_k$ and $S_{\alpha} := T_{\alpha}^* T_{\alpha} = I_{\mathcal{H}}$. In this setting, one may think of T_{α} as adding an empty skeleton in the lowest possible position with a single open node on the left or on the right depending on whether k is in I or J , and S_{α} as applying T_{α} and then immediately closing

the node.



To simplify notation, let $C : \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H}$ be the concatenation operator

$$C(\xi_1, \xi_2) = \xi_1 \otimes \xi_2$$

(where, if either ξ is Ω , the other vector is returned) which is clearly well-defined (and extends by linearity). Further, let $\Sigma : \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H}$ be defined by $\Sigma(\xi, \Omega) = \xi = \Sigma(\Omega, \xi)$ for all $\xi \in \mathcal{H}$ and, for $\alpha : \{1, \dots, n\} \rightarrow I \sqcup J$ and $\beta : \{1, \dots, m\} \rightarrow I \sqcup J$, we define

$$\Sigma(f_1 \otimes \dots \otimes f_n, f_{n+1} \otimes \dots \otimes f_{n+m}) := \sum_{\sigma} f_{\sigma(1)} \otimes \dots \otimes f_{\sigma(n+m)},$$

where the sum is over all permutations $\sigma \in S_{n+m}$ which preserve the order of $\{1, \dots, n\}$ and preserve the order of $\{n+1, \dots, n+m\}$. For example, for $1, 2, 3, 4 \in I \sqcup J$,

$$\begin{aligned} \Sigma(e_1 \otimes e_2, e_3 \otimes e_4) &= e_1 \otimes e_2 \otimes e_3 \otimes e_4 + e_1 \otimes e_3 \otimes e_2 \otimes e_4 + e_3 \otimes e_1 \otimes e_2 \otimes e_4 \\ &\quad + e_1 \otimes e_3 \otimes e_4 \otimes e_2 + e_3 \otimes e_1 \otimes e_4 \otimes e_2 + e_3 \otimes e_4 \otimes e_1 \otimes e_2. \end{aligned}$$

We will only use Σ in the case that the first coordinate is a tensor of basis vectors with indices from I and the second coordinate is a tensor of basis vectors with indices from J . Roughly Σ will be used to construct skeletons where one needs to interleave nodes on the left and right, and C will be used to join parts of the skeleton in the correct order.

For $\alpha : \{1, \dots, n\} \rightarrow I \sqcup J$ we define S_α and T_α on Ω by

$$T_\alpha(\Omega) = L_{\alpha(n)} L_{\alpha(n-1)} \dots L_{\alpha(1)}(\Omega)$$

and

$$S_\alpha(\Omega) = L_{\alpha(n-1)} L_{\alpha(n-2)} \dots L_{\alpha(1)}(\Omega).$$

(Note that these correspond to the starter and empty skeleton of α , respectively.)

We will now define S_α and T_α for $n \geq 2$ on a dense subset of \mathcal{H} when $\alpha(n) \in I$. For $\beta : \{1, \dots, m\} \rightarrow J$ and

$$\eta = e_{\beta(1)} \otimes \dots \otimes e_{\beta(m)} \in \mathcal{H}.$$

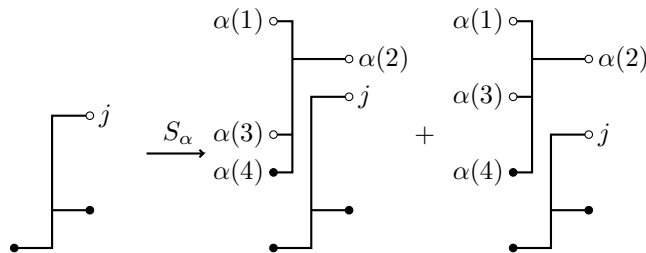
One should think of this as a partially completed skeleton where all open nodes are on the right) let k be the largest element of $\{1, \dots, n\}$ such that $\alpha(k) \in J$ (or $k = 0$ if α maps into I). We define $S_\alpha(\eta)$ to be

$$C(\Sigma(e_{\alpha(n-1)} \otimes \dots \otimes e_{\alpha(k+1)}, e_{\beta(1)} \otimes \dots \otimes e_{\beta(m)}), e_{\alpha(k)} \otimes \dots \otimes e_{\alpha(1)}).$$

Again, one should think of this as the sum of all valid partially completed skeletons where the old skeleton is below and to the right of starter skeleton corresponding to α . For example, if $\alpha : \{1, 2, 3, 4\} \rightarrow I \sqcup J$ satisfies $\alpha^{-1}(I) = \{1, 3, 4\}$ and $\alpha^{-1}(J) = \{2\}$, and $j \in J$, then

$$S_\alpha e_j = e_{\alpha(3)} \otimes e_j \otimes e_{\alpha(2)} \otimes e_{\alpha(1)} + e_j \otimes e_{\alpha(3)} \otimes e_{\alpha(2)} \otimes e_{\alpha(1)}.$$

This action may correspond to the following diagrams:



It may appear problematic that multiple diagrams appear, but the diagrams that survive in the end will depend on whether the annihilations at the appropriate stages are from I (left nodes) or from J (right nodes). In addition, we let

$$T_\alpha(\eta) := L_{\alpha(n)}(S_\alpha(\eta)).$$

Finally, suppose $\beta : \{1, \dots, m\} \rightarrow I \sqcup J$ and $k \in \{1, \dots, m\}$ are such that $\beta(t) \in I$ for $t = 1, \dots, k-1$ and $\beta(k) \in I$, and

$$\eta = e_{\beta(1)} \otimes \dots \otimes e_{\beta(m)} \in \mathcal{H}.$$

This corresponds to a partially completed skeleton with open nodes on both the left and right where the lowest open node on the left is the k^{th} open node from the bottom. We set $S_\alpha(\eta) = 0$ if there exists a $t \in \{1, \dots, n\}$ such that $\alpha(t) \in J$: since the partially completed skeleton has open nodes on the left and right we cannot add the starter skeleton of α without introducing a crossing. Otherwise $\alpha(t) \in I$ for all $t \in \{1, \dots, n\}$ and we set

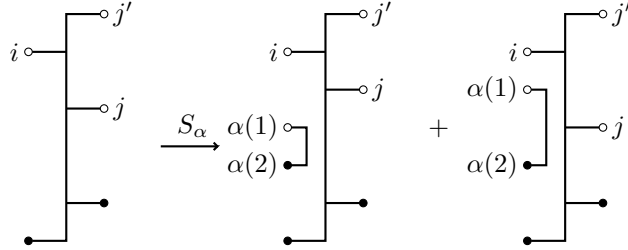
$$S_\alpha(\eta) := C(\Sigma(e_{\alpha(n-1)} \otimes e_{\alpha(n-2)} \otimes \dots \otimes e_{\alpha(1)}, e_{\beta(1)} \otimes \dots \otimes e_{\beta(k-1)}), e_{\beta(k)} \otimes \dots \otimes e_{\beta(m)}).$$

One can think of this as the sum of all valid partially completed skeletons where the starter skeleton of α sits below the lowest open node on the left of the old skeleton.

For example, if $\alpha : \{1, 2\} \rightarrow I$ and $i \in I, j, j' \in J$ then

$$S_\alpha e_j \otimes e_i \otimes e_{j'} = e_{\alpha(1)} \otimes e_j \otimes e_i \otimes e_{j'} + e_j \otimes e_{\alpha(1)} \otimes e_i \otimes e_{j'}.$$

This action may correspond to the following diagrams:



Again we let

$$T_\alpha(\eta) := L_{\alpha(n)}(S_\alpha(\eta)).$$

As S_α and T_α have been defined on an orthonormal basis, we may extend them by linearity to obtain densely defined operators on \mathcal{H} . Note that these operators may not be bounded due to the action of Σ . However, note that if $\alpha : \{1, \dots, n\} \rightarrow I$ then S_α and T_α act on the Fock subspace generated by $\{e_i\}_{i \in I}$ as $L_{\alpha(n-1)} \dots L_{\alpha(1)}$ and $L_{\alpha(n)} \dots L_{\alpha(1)}$ respectively. Thus these operators correspond to those used in Nica's operator model.

We define S_α and T_α where $\alpha(n) \in J$ in a similar manner, interchanging I and J as necessary.

6.4. The operator model for pairs of faces. With Construction 6.3.1, the operator model for a pair of faces is at hand.

Theorem 6.4.1. *Let $z = (\{z_i\}_{i \in I}, \{z_j\}_{j \in J})$ be a pair of faces in a non-commutative probability space (\mathcal{A}, φ) . With notation as in Construction 6.3.1, consider the (unbounded) operator*

$$\Theta_z := I + \sum_{n \geq 1} \sum_{\alpha: \{1, \dots, n\} \rightarrow I \sqcup J} \kappa_\alpha(z) T_\alpha.$$

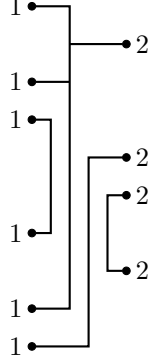
For $k \in I \sqcup J$ define

$$Z_k := L_k^* \Theta_z = L_k^* + \sum_{n \geq 0} \sum_{\alpha: \{1, \dots, n+1\} \rightarrow I \sqcup J, \alpha(n+1)=k} \kappa_\alpha(z) S_\alpha.$$

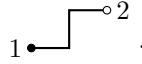
Then, if $T \in \text{alg}(\{Z_k\}_{k \in I \sqcup J})$ then $\langle T\Omega, \Omega \rangle$ is well-defined. Moreover, if $\psi(T) = \langle T\Omega, \Omega \rangle$, the joint distribution of $\{Z_k\}_{k \in I \sqcup J}$ with respect to ψ is the same as the joint distribution of z with respect to φ .

Before we begin the proof, we give the following example.

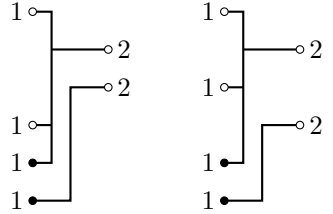
Example 6.4.2. In this example, let $I = \{1\}$ and $J = \{2\}$. We will examine how the completed skeleton below is constructed for $Z_1 Z_2 Z_1 Z_1 Z_2 Z_2 Z_1 Z_2 Z_1 Z_1$.



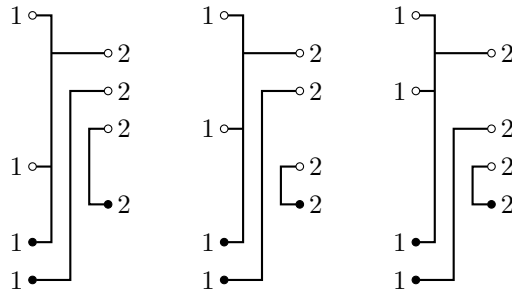
First $\kappa_{(21)}(z)S_{(21)}$ is applied to get the partially completed skeleton



Then $\kappa_{(1211)}(z)S_{(1211)}$ is applied to obtain the following collection of partially completed skeletons

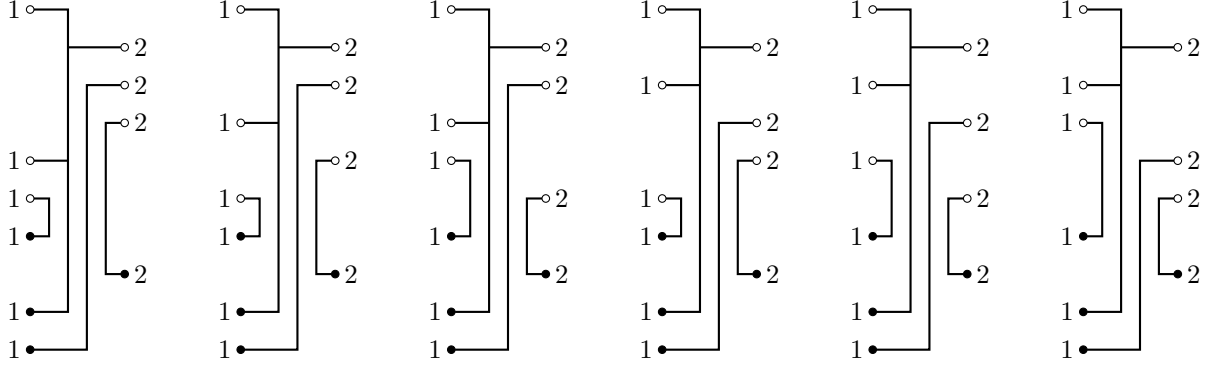


Applying $\kappa_{(22)}S_{(22)}$ then gives the following collection of partially completed skeletons (where the first two below are from the first above and the third below is from the second above)

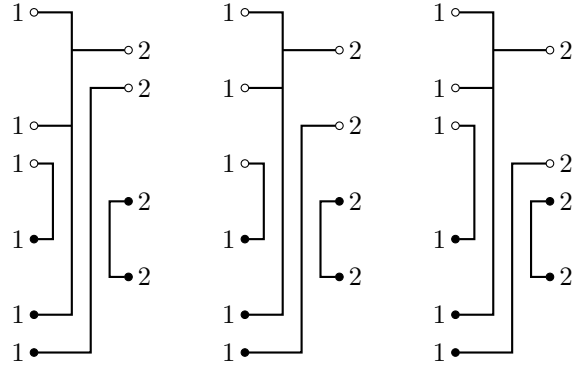


and applying $\kappa_{(11)}S_{(1)}$ then gives the following collection of partially completed skeletons (where the first below is from the first above, the second and third below are from the second above, and the last three are

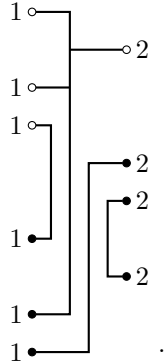
from the third above).



Applying L_2^* then gives the following collection of partially completed skeletons (where the first, second, and fourth diagrams above were destroyed)



and applying L_2^* kills all but the last diagram to give



Applying $L_1^* L_2^* L_1^* L_1^*$ then gives us the desired diagram. We also see the diagram was weighted by

$$\kappa_{(21)}(z) \kappa_{(1211)}(z) \kappa_{(22)} \kappa_{(11)}(z)$$

which is the correct product of bi-free cumulants for this bi-non-crossing partition.

Proof of Theorem 6.4.1. Let $\alpha : \{1, \dots, n\} \rightarrow I \sqcup J$. To see that

$$\psi(Z_{\alpha(1)} \cdots Z_{\alpha(n)}) = \varphi(z_{\alpha(1)} \cdots z_{\alpha(n)}),$$

we must demonstrate that the sum of over all

$$T_{\alpha(k)} \in \{L_{\alpha(k)}^*\} \cup \{\kappa_{\beta}(z) S_{\beta} \mid \beta : \{1, \dots, m\} \rightarrow I \sqcup J, \beta(m) = \alpha(k)\}$$

of

$$\langle T_{\alpha(1)} \cdots T_{\alpha(n)} \Omega, \Omega \rangle$$

is precisely $\varphi(z_{\alpha(1)} \cdots z_{\alpha(n)})$. By construction $T_{\alpha(1)} \cdots T_{\alpha(n)}$ acting on Ω corresponds to creating a (sequence of) partially completed skeletons and $\langle T_{\alpha(1)} \cdots T_{\alpha(n)} \Omega, \Omega \rangle$ will be the weight of the skeleton if the skeleton is complete and otherwise will be zero. Since

$$\varphi(z_{\alpha(1)} \cdots z_{\alpha(n)}) = \sum_{\pi \in BNC(\alpha)} \kappa_{\pi}(z).$$

it suffices to show that there is a bijection between completed skeletons and elements π of $BNC(\alpha)$ under the condition that the weight of the skeleton is $\kappa_{\pi}(z)$.

To see that every completed skeleton is in $BNC(\alpha)$, note that if a S_{β} appears in the k^{th} position of the product, a closed node labelled $\alpha(k)$ appears $n - k + 1$ nodes from the bottom of the skeleton and if $L_{\alpha(k)}^*$ appears in the k^{th} position of the product, a closed node labelled $\alpha(k)$ must appear as the $n - k + 1$ from the bottom (as later actions do not modify closed nodes). In either case, the node $n - k + 1$ from the bottom in a final skeleton must be labelled $\alpha(k)$. Moreover, since S_{β} creates one closed node (corresponding to the value β takes on the largest element of its domain) and $m - 1$ open nodes, and since $L_{\alpha(k)}^*$ changes one open node to a closed node that must be labelled $\alpha(k)$, we see that each completed skeleton must be in $BNC(\alpha)$.

To see that each element π of $BNC(\alpha)$ is created by precisely one product, examine the blocks of π . If V is a block of π , we see that if the labels of the nodes corresponding to V are $\beta : \{1, \dots, m\} \rightarrow I \sqcup J$ read from the top down, then V must be created by S_{β} occurring in the k^{th} position in the product where the lowest node of V occurs as the node $n - k + 1$ from the bottom. It is then apparent that by placing annihilation operators in the remaining positions, we may construct π in a way that the weight of the final completed skeleton is precisely the product of the appropriate bi-free cumulants. To see that we do not obtain additional completed skeletons for this product, we note that when Σ is used for an S_{β} , we allow all possible mixing of the open nodes on the left and on the right. Thus, after applying k terms of the product to Ω , we obtain all partially completed skeleton with the same bottom k closed nodes and the same type of blocks connected to these closed nodes (see Example 6.4.2). However each annihilation operator chooses whether any mixing should have created an open node on the left or on the right at the corresponding point in the product and thus the sequence of annihilation operators completely determines only one non-zero mixing. \square

Remark 6.4.3. In Theorem 7.4 of [5], an operator model for the bi-free central limit distributions was given as sums of creation and annihilation operators on a Fock space. It is interesting that the operator model from Theorem 6.4.1 uses different operators. Indeed for $i, i' \in I$ and $j \in J$, one can see

$$S_{(i,i')} = \sum_{n \geq 0} \sum_{\alpha: \{1, \dots, n\} \rightarrow J} L_{\alpha(1)} \cdots L_{\alpha(n)} L_i L_{\alpha(n)}^* \cdots L_{\alpha(1)}^*$$

and

$$S_{(j,i')} = R_j P$$

where P is the projection onto the Fock subspace of \mathcal{H} generated by $\{e_k\}_{k \in J}$ and R_j is the right creation operator corresponding to e_j . Therefore, if $c_{k_1, k_2} = \varphi(z_{k_1} z_{k_2})$ for $k_1, k_2 \in I \sqcup J$ with z a bi-free central limit distribution, Theorem 6.4.1 produces the operators

$$Z_k = L_k^* + \sum_{k' \in I \sqcup J} c_{k', k} S_{(k', k)}$$

which are very different operators that $L_k + L_k^*$ (if $k \in I$) and $R_k + R_k^*$ (if $k \in J$). The main issues with the model involving $\{L_i, L_i^*, R_j, R_j^* \mid i \in I, j \in J\}$ is that the vectors obtained by applying the algebra generated by these operators to Ω do not generate the full Fock space - indeed they only generate vectors of the form

$$e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_{j_m} \otimes \cdots \otimes e_{j_1}$$

where $n, m \geq 0$, $i_1, \dots, i_n \in I$, and $j_1, \dots, j_m \in J$. It is not difficult to see that the vectors obtained by the algebra generated $\{L_i^*, L_j^*, S_{(i,i)}, S_{(j,j)} \mid i \in I, j \in J\}$ applied to Ω generate the full Fock space.

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